$$D = \sqrt{S^2 - (P^2 + Q^2)} \quad [VAD]$$

#14 Time-varying state of electric circuits

Equations of el. circuits under time-varying conditions

$$\sum_{k \in (a)} i_k = 0, \quad a = \overline{1, N-1}$$

$$\sum_{k \in (p)} u_{tk} = 0, \quad p = \overline{1, B-N+1}$$

Eq. offered by Kirchhoff's and Joubert's theorems

$$u_{tk} = u_{Rk} + u_{Ck} + u_{Lk} - e_k - u_k = u_{Rk} + u_{Ck} + \frac{d\phi_k}{dt} - e_k - u_k,$$

where u_{Ck} at the terminals of a capacitor is linked with the current i_{Ck} through the capacitor by equation

$i_{Ck} = \frac{dq_k}{dt}$, may be combined to reduce the system of 2B equations with 2B possibly unknown quantities to a simpler system of B eq. with B possibly unknown quantities - one for every branch of the circuit.

- lumped circuit operating under time varying conditions = dynamic circuit
- given data for a dynamic circuit: circuit's configuration, $R_k, C_k, L_{kj}, e_k(t), a_k(t)$ of ideal circuit elements that are present
- we have to find out the values of the electric quantities characterising the state of every ideal el. of the circuit: the current/voltage at every resistor's / coil's terminals, the voltage at every capacitor's terminals, the current

through every voltage generator and the voltage at the terminals of every current generator.

- the nr. of these possibly unknown, obv. greater than B (nr. of branches), can be reduced to exactly B unk. quantities, one for every branch: the current i_K along every branch without a capacitor or an ideal current generator, the voltage u_{CK} at the terminals of every capacitor in a branch without any ideal current generators, and the voltage u_K at the terminals of every current generator.

The current i_K going through a resistor, coil or ideal voltage generator can be obtained from the voltage at the terminals of a capacitor (when it is present on the branch):

$$\boxed{i_K = C_K \cdot \frac{du_{CK}}{dt}}$$

If the capacitor is present on a branch, along with an ideal current generator, the voltage at its terminals can be derived as:

$$\boxed{u_{CK} = u_{CK0} + \frac{1}{C_K}\int_{t_0}^{t} a(t)\,dt}$$

If the voltage at the terminals of reactive el. are detailed, the explicit system of Kirchhoff's eq. are:

$$\begin{cases} \sum\limits_{K\in(p)}\left(u_{RK} + u_{CK} + \dfrac{d\Phi_K}{dt} - u_K\right) = \sum\limits_{K\in(p)} e_K \,, \quad p = \overline{1, B-N+1} \\ \sum\limits_{K\in(a)} i_K = 0 \,, \quad a = \overline{1, N-1} \end{cases}$$

$$i_{CK} = \frac{d g_K(u_{CK})}{dt}$$

In the case of a lumped linear circuit, the eq. are:

$$\begin{cases} \sum_{K\in(p)} \left(R_K i_K + u_{CK} + \sum_{S=1}^{B} L_{KS} \frac{d i_S}{dt} - u_K \right) = \sum_{K\in(p)} e_K, \quad p=\overline{1, B-N+1} \\ \sum_{K\in(a)} i_K = 0, \quad a = \overline{1, N-1} \end{cases}$$

$$i_{CK} = \frac{d g_K}{dt} = C_K \cdot \frac{d u_{CK}}{dt}$$

The explicit system of Kirchhoff's eq. for the characterization of a lumped circuit having B branches and N nodes is made up of B eq. with B unk. quantities.

$$u_{LK} = \frac{d\phi_K}{dt}, \quad i_{CK} = \frac{d g_K}{dt}$$

In part., in the case of linear lumped circuits, the explicit syst. of Kirchhoff's eq. is a syst. of inhom. linear diff. eq. with constant coefficients.

$$u_{LK} = \sum_{S=1}^{B} L_{KS} \frac{d i_{LS}}{dt}, \quad i_{CK} = C_K \frac{d u_{CK}}{dt}$$

- a "proper tree" contains all the capacitors and voltage generators and leaves all the coils & current generators to be placed in the chords of the associated co-tree.
- a "proper circuit" contains at least a proper tree, or, equivalently is characterized by 2 cond.:
  1) there's no loop containing only independent ideal voltage generators and ideal capacitors
  2) there aren't any cut-sets that contain (or nodes where concur) branches consisting exclusively of indepen-

dent ideal current generators and ideal coils.
- cut-sets = nodes

The solution to the eq. of linear dynamic circuits (part 1)

The eq. which describe a linear dynamic circuit are:

$$\begin{cases} \sum_{K\in(p)} \left(R_K i_K + u_{CK} + \sum_{s=1}^{B} L_{Ks} \frac{di_s}{dt} - u_K \right) = \sum_{K\in(p)} e_K, \quad p = \overline{1, B-N+1} \\ \sum_{K\in(a)} i_K = 0, \quad a = \overline{1, N-1} \end{cases}$$

$$i_{CK} = C_K \frac{du_{CK}}{dt}$$

Obs.: I Kirchhoff's voltage eq. are linear diff. inhom. eq.: they are linear, obviously include derivatives of some of the unknown quantities, and they also include some given time-dependent voltages in the right-hand side.

II Kirchhoff's current eq. are generally linear diff. inhom. eq.: they are linear and may include diff. terms as $i_K \to i_{CK} = C_K \frac{du_{CK}}{dt}$ and given time dependent terms as $i_K \to a_K(t)$ corresponding to the presence of such el. - capacitors or current generators - in the branches concurring in the node under consideration.

III All the coeff. of the unk. quantities (or their derivatives) are constant. However, it often happens that the circuit's components (resistances, capacitors, coils) are changed. Therefore, we should take into account the possible

by the operation of some switches

situations when the coeff. of the unk. quantities or their derivatives may be cont. over specified time intervals, separated by moments of finite discont.

The prob. associated with the evolution of a linear dynamic circuit $\Leftrightarrow$ the prob. of solving a system of linear diff. inhom. eq. with constant coeff. over a specified time interval, usually taken as $t \in [0, T)$, or even as $t \in [0, \infty)$.

Reformulated eq. in a so called canonical form:

$$\left\{ \frac{df_K}{dt} + \sum_{s=1}^{m} a_{Ks} f_s = g_K(t), \quad K = \overline{1, m} \right.$$

The sol. of a canonical system of linear diff. inhom. eq. with constant coeff. <u>exists</u> and is <u>cont.</u> on the closed interval of continuity the coeff. and given right-hand side time-fctions, and is <u>derivable</u> on the open int. of cont. of coeff. and given right-hand side time-fctions. Under these conditions, the sol. is <u>uniquely det.</u> if the value of every unk. fction is specified at a so called initial moment t_0 in the int. of cont. of coeff. and given right-hand side time-dep. fctions.

The sol. for any unk. time-dependent fction $f_K(t)$ (unk. voltage u_K or current i_K) can be decomposed into a so called "<u>free sol.</u>" and a so called "<u>forced sol.</u>",

$f_K(t) = f_K^0(t) + f_K^F(t)$.

The free sol. $f_K^0(t)$ = the general sol. of the hom. system of diff. eq., obtained from the original (complete) syst. of eq. if all given time-dependent fctions (in the right-hand side) are equalled to zero, (continued)

The sol. to the eq. of linear dynamic circuits (part 2)

$$\left\{ \frac{d f_k^0}{dt} + \sum_{s=1}^{m} a_{ks} f_s^0 = 0, \quad k = \overline{1,m} \right.$$

This free sol. depends not only on time, but also on some yet undetermined m integration constants,

$$f_k^0 = f_k^0 (t, A_1, \dots, A_m)$$

The time-dependence of the free sol. is of the exponential kind. The gen. form of the sol. to a linear hom. diff. equation involving a fction and its 1st derivative,

$$\frac{df}{dt} + af = 0,$$

is an exponential fction, $f = Ae^{rt}$.

A - to be det. from other considerations

r - the coeff. of time variable, and can be obt. by a simple substitution:

$$rAe^{rt} + aAe^{rt} = 0 \Rightarrow r = -a$$

In the gen. case of the canonical syst. of lin. diff. hom. eq. with ct coeff., such substitutions $f_k = A_k e^{rt}$ yield:

$$\left\{ rA_k e^{rt} + \sum_{s=1}^{m} a_{ks} A_s e^{rt} = 0, \quad k = \overline{1,m} \right. \Leftrightarrow$$

$$\Leftrightarrow \left\{ rA_k + \sum_{s=1}^{m} a_{ks} A_s = 0, \quad k = \overline{1,m} \right. \longrightarrow$$ its sol. is non-trivial iff the system's det. $= 0$

$$\begin{vmatrix} r+a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & r+a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & r+a_{mm} \end{vmatrix} = 0$$

This so called <u>characteristic eq.</u>, associated with the considered syst. of diff. eq., is an algebraic eq. of the m^{th} degree, having therefore m simple/multiple, real/complex roots r.

The free sol. for any unk. fction $f_K(t)$ is a linear combination of exponential fctions.

The forced sol. $f_K^F(t)$ is a <u>particular sol. of the inhom. (complete) syst. of eq.</u>,

$$\left\{ \frac{df_K^F}{dt} + \sum_{s=1}^{m} a_{Ks} f_s^F = g_K(t), \quad K = \overline{1,m} \right.$$

There's no general rule referring to the time-dependence of the forced sol. However, if the given fction characterising the generators ($e(t)$, $a(t)$) are exponential fctions, then the forced sol. is a fction of the same kind (exponential).

Time-dependent fctions representing the sol. of a linear dynamic circuit have the following gen. form:

$$f_K(t) = f_K^0(t, A_1, \dots, A_m) + f_K^F(t)$$

→ cont. & der. on the closed, resp. open, interval of continuity of eq. coeff. and given right-hand side fctions characterising the generators

Det. the values of the integration constants $A_1, \dots A_m$:

- we're taking into consideration a node "a"

$\sum_{k\in(a)} i_k = 0$ → we rewrite Kirchhoff's eq. for this node by separating the currents carried by the branches that include capacitors from those that don't

$$\sum_{k\in(a)} i_{Ck} + \underbrace{\sum_{k\in(a)} i_k}_{\text{no } C} = 0$$

$$\sum_{k\in(a)} i_{Ck} = \sum_{k\in(a)} \frac{dq_k}{dt} = \frac{d}{dt}\sum_{k\in(a)} q_k = \frac{d}{dt} q_{(a)}, \text{ where}$$

$q_{(a)} = \sum_{k\in(a)} q_k$ = total charge associated with the node "a"

$$\sum_{k\in(a),\text{ no } C} i_k = i_{(a)\text{non } C}$$

$$\Rightarrow i_{(a)\text{non } C} = -\frac{d}{dt} q_{(a)}$$

Necessary cond. for a consistent sol.:

$$q_{(a)}\Big|_{t=0_-} = q_{(a)}\Big|_{t=0+}$$

- the right-hand side = the limit of the node charge when $t\to 0, t>0$
- the left-hand side has to be given!

Necessary continuity cond. in terms of the circuit's variables:

$$\sum_{k\in(a)} C_k u_{Ck0} = \sum_{k\in(a)} C_k u_{Ck}(0+, A_1, \dots, A_m), \text{ where}$$

$u_{CK0} = u_{CK}(0-)$ is necessarily given data.

- we consider a loop "p"

$$\sum_{K\in(p)} u_K = 0$$

- we rewrite the above eq. by separating the voltages across coils from those across any non-inductive el. of the circuit

$$\sum_{K\in(p)} u_{LK} + \sum_{K\in(p), \text{no } L} u_K = 0$$

$$\sum_{K\in(p)} u_{LK} = \sum_{K\in(p)} \frac{d\phi_K}{dt} = \frac{d}{dt}\sum_{K\in(p)} \phi_K = \frac{d}{dt}\phi(p)$$

$\phi_{(p)}$ = total flux of the "p" loop

$$\sum_{K\in(p), \text{no } L} u_K = u_{(p)\,\text{non } L}$$

$$\Rightarrow u_{(p)\,\text{non } L} = -\frac{d}{dt}\phi_{(p)}$$

Necessary cond. for a consistent sol.:

$$\phi_{(p)}\Big|_{t=0-} = \phi_{(p)}\Big|_{t=0+}$$

- the right-hand side = the limit of the loop flux when $t\to 0$, $t>0$.
- the left-hand has to be given

Necessary continuity cond. when expressing the loop flux in terms of the circuit's variables:

$$\sum_{K\in(p)}\sum_{S=1}^{B} L_{KS}\, i_{LS0} = \sum_{K\in(p)}\sum_{S=1}^{B} L_{KS}\, i_{LS}(0+, A_1, \dots, A_n)$$

$i_{LS0} = i_{LS}(0-)$ is necessarily given data.

Algorithm of det. the evolution of a linear dynamic circuit:

1) det. the initial values $u_{CK0} = u_{CK}(0-)$ and $i_{LS0} = i_{LS}(0-)$ characterising the initial state of all reactive circuit el.

2) write Kirchhoff's eq. which describe the dynamic circuit, and reduce them in a syst. of a nr. of inhom. linear diff. eq. equal to the nr. of reactive circuit el. in the circuit

3) write the corresponding syst. of hom. linear diff. eq. for the free sol., construct and solve the associated characteristic equation for the coefficients of the time variable t in the exponent of the exponential kind of free sol. Ae^{rt}, thus determining the free sol. of the system.

4) det. the forced sol. of the complete system of inhom. linear diff. eq. of the dynamic circuit

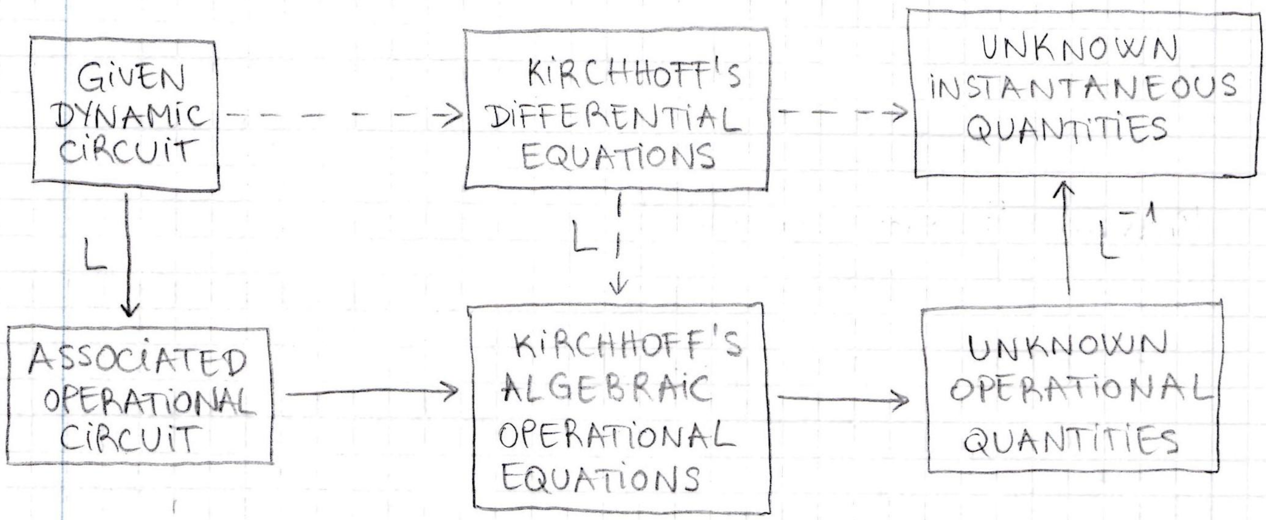
5) assemble the total sol. of the circuit's equations as the sum of free and forced solutions and impose the initial conditions of continuity to construct a syst. of eq. for the calc. of the integration constants, and thus completely determine the sol.

Obs.: The free sol. of the equations of ac ______ ? circuits, where there are no cut-sets of zero conductance or loops of zero resistance exponentially decaying to zero, such that it remains the forced solution only. The decaying process is characterized by the time constants as:

$$\tau_j = \frac{1}{|Re\{r_j\}|};$$

the remaining forced sol. being practically reached after a delay $\Delta t \approx (3 \ldots 5) \cdot \max_j \tau_j$

#16 Operational calculus of linear dynamic circuits (Laplace method)



Laplace transformations are associated to the original circuit and to its instantaneous quantities, resulting in a corresponding operational circuit.

The direct and inversed Laplace transformations relate time-varying fctions $f(t)$, called original fctions, and complex fctions of the complex variable $F(s)$, called image fctions.

One-sided Laplace transformation

An original fction $f: \mathbb{R} \to \mathbb{R}$ satisfying the following cond.:

1) f sectionally smooth on its support (meaning that its support is separable into a finite nr. of cont. int. of monotony, separated by a corresponding finite

nr of points of discont.)

2) f exhibits at most an <u>exponential rate of increase</u>
$\exists M > 0, \sigma_0, t_0 \geq 0$ such that $|f(t)| < Me^{\sigma_0 t}$ if $t > t_0$

3) f's support is restricted to the non-negative values of the time variable $f(t) = 0$ for $t < 0$

<u>Direct Laplace transf.</u> of an original fction:

$$F(s) = L\{f(t)\} = \int_{0-}^{\infty} f(t)\, e^{-st}\, dt, \quad \text{where } s = \sigma + j\omega \in \mathbb{C}$$

<u>Inversed Laplace transf.</u> (computed according to Mellin-Fourier formula):

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st}\, ds$$

where the integration in the complex plane is performed along the straight line $Re\{s\}, \sigma \geq \sigma_0$, parallel to the imaginary axis, from $\sigma - j\infty$ to $\sigma + j\infty$.

<u>Properties of Laplace transf.</u>:

1) linearity

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

2) the Laplace transf. of the derivative of an original fction is

$$L\left\{\frac{df(t)}{dt}\right\} = sL\{f(t)\} - f(0-)$$

- in particular: $L\left\{\frac{df(t)}{dt}\right\} = sL\{f(t)\}$ if $f(0-) = 0$

3) the Laplace transf. of the integral of an original fction is

$$L\left\{\int_0 f(\theta)\,d\theta\right\} = \frac{1}{s} L\{f(t)\}$$

4) The Laplace transf. of an original fction with a delayed argument (supposted to be approaching 0 for neg. values of the argument)

$$L\{f(t-\tau)\} = e^{-st} L\{f(t)\}$$

5) The Laplace transf. of the product of an original fction by an exponential fction

$$L\{f(t)\} = F(s) \Rightarrow L\{f(t)e^{at}\} = F(s-a),\ a \in \mathbb{C}$$

6) The Laplace transf. of an original fction with a scaled argument $(a>0)$

$$F(s) = L\{f(t)\} \Rightarrow L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

7) The limit values of the original fction are related to the limit values of the image fction according to:

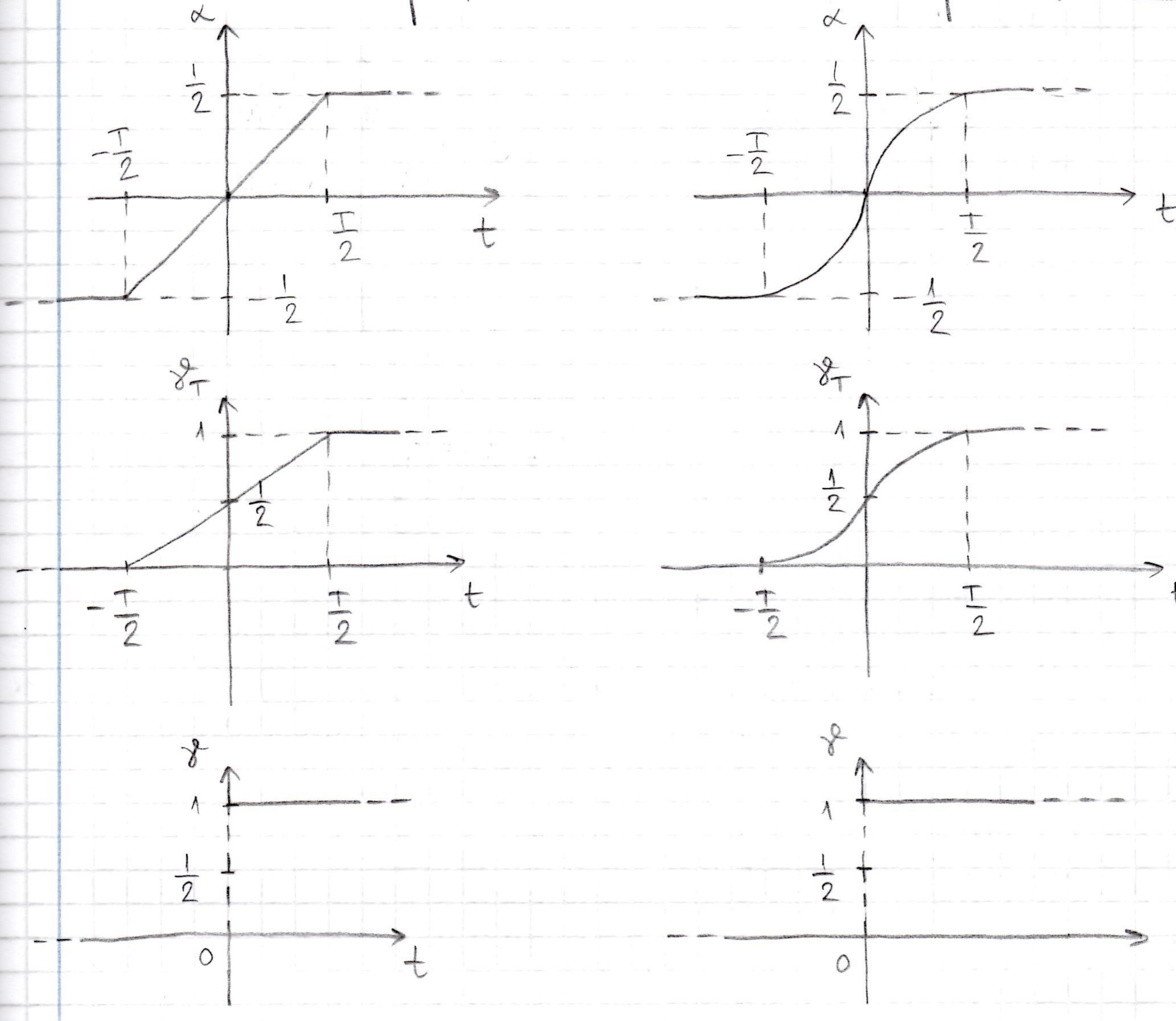
$$L\{f(t)\} = F(s) \Rightarrow \begin{cases} f(0+) = \lim\limits_{t\to 0+} f(t) = \lim\limits_{s\to\infty} sF(s) \\ f(\infty) = \lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} sF(s) \end{cases}$$

8) The Laplace transf. of the convolution product of 2 original fctions is the alg. product of their image fctions (Borel's theorem)

$$\left.\begin{aligned} F(s) &= L\{f(t)\} \\ G(s) &= L\{g(t)\} \end{aligned}\right| \Rightarrow F(s)G(s) = L\{(f*g)(t)\} =$$

$$= L\left\{\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right\}$$

Unit-step fction $\gamma(t)$ & unit-impulse fction $\delta(t)$



The unit-step (Heaviside) fction $\gamma(t)$ (sometimes $\sigma(t)$ or $1(t)$):

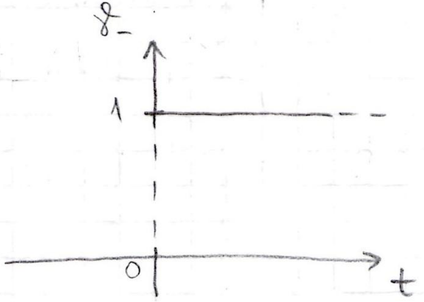
Let us consider $\alpha(t)$ a fction which is cont., mon., odd, satisfying the cond. $\alpha(0)=0$, $|\alpha(t)| = \frac{1}{2}$ for $|t| > \frac{T}{2}$.

Let $\gamma_T(t) = \alpha(t) + \frac{1}{2}$

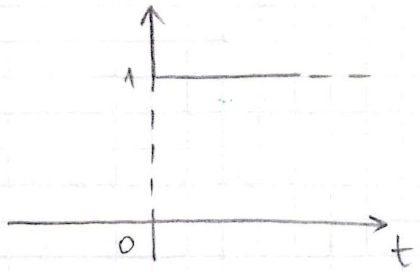
If we consider the lim of $\gamma_T(t)$ as $T \to 0$, then the unit-step fction is defined as:

$$\gamma(t) = \lim_{T\to 0} \gamma_T(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}, & t = 0 \\ 1, & t > 0 \end{cases}$$

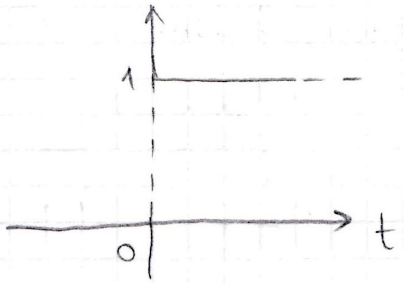
Simplified Heaviside fctions:



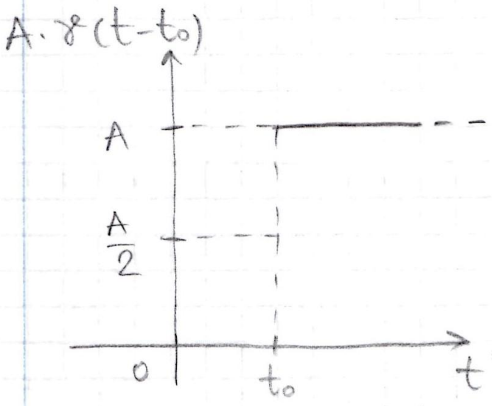
$$\gamma_-(t) = \begin{cases} 0, & t \le 0 \\ 0, & t > 0 \end{cases}$$



$$\gamma_+(t) = \begin{cases} 0, & t < 0 \\ 0, & t \ge 0 \end{cases}$$

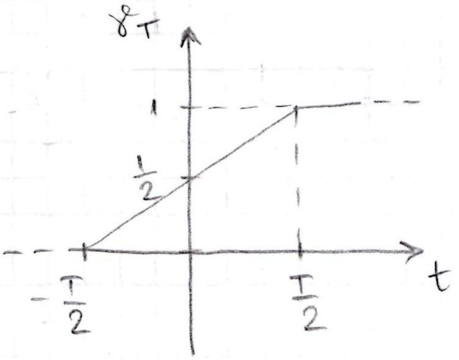


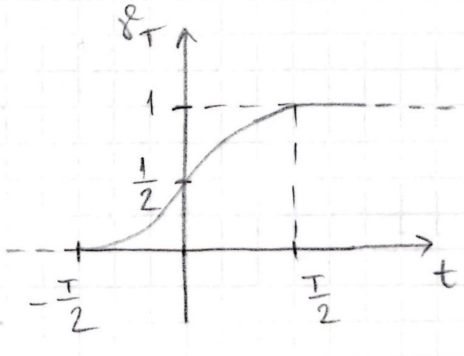
$$\gamma_0(t) = \begin{cases} 0, & t < 0 \\ 0, & t > 0 \end{cases}$$

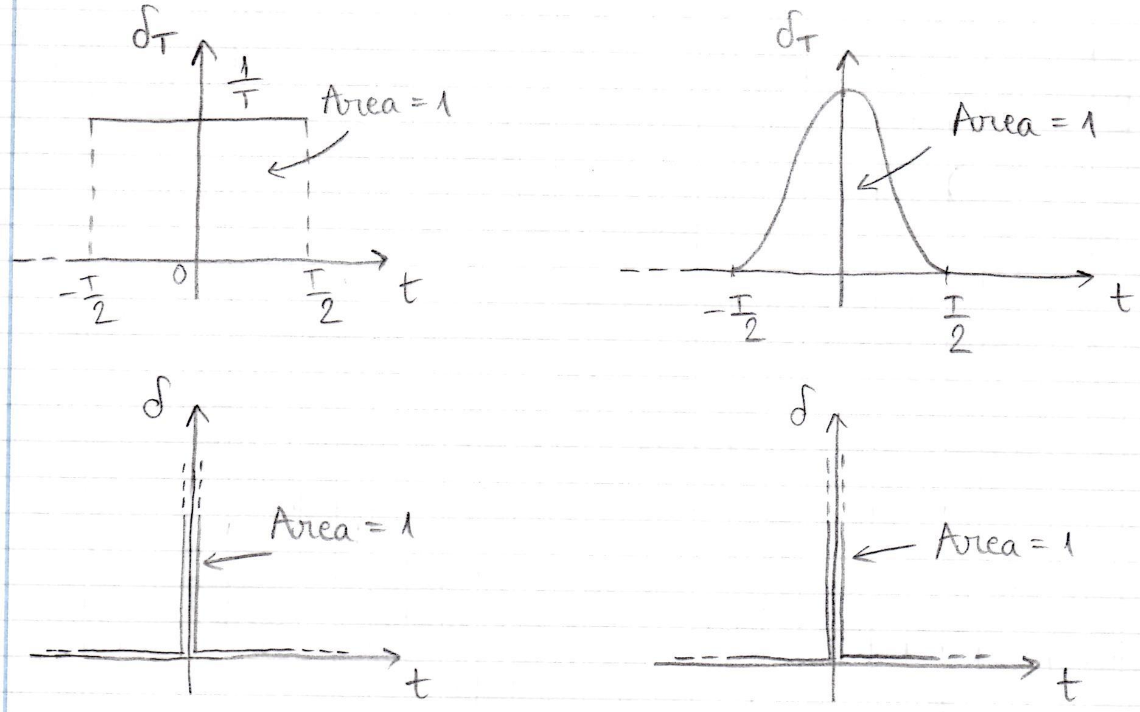


→ step variation of amplitude A occuring at a moment t_0

$$A\gamma(t-t_0) = \begin{cases} 0, & t < t_0 \\ A/2, & t = t_0 \\ A, & t > t_0 \end{cases}$$







The unit-impulse (Dirac) fction

Let's consider the finite-support unit-impulse fction $\delta_T(t) = \frac{d}{dt}\gamma_T(t)$, as drawn in the above figures for the 2 versions of the fction.

$$T \rightarrow 0 \rightarrow \delta(t) = \lim_{T\rightarrow 0} \delta_T(t) = \lim_{T\rightarrow 0} \frac{d}{dt}\gamma_T(t)$$

Properties:

$\delta(t) = 0$ for $t \neq 0$

$\delta(0)$ not defined

$\int_{-\infty}^{+\infty} \delta(t)\,dt = \int_{-\varepsilon}^{+\varepsilon} \delta(t)\,dt = 1$ for vanishingly small $\varepsilon > 0$

$\int_{-\infty}^{+\infty} f(t)\,\delta(t)\,dt = \int_{-\varepsilon}^{+\varepsilon} f(t)\,\delta(t)\,dt = f(0)$ for any fction $f(t)$ cont. at $t=0$

The unit-impulse fction is useful in expressing concisely a finite fction contents concentrated at some point $t = t_0$ in an integral sense. Concentrated contents C occuring at a moment t_0 is represented as $C\delta(t-t_0)$ in the sense that

$$\int_{-\infty}^{+\infty} C\delta(t-t_0)\,dt = C\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t-t_0)\,dt = C$$

Important illustrative examples of Laplace transf. of usual time-dependent original fctions:

$$\left.\begin{array}{l} f(t): \mathbb{R} \to \mathbb{R} \\ f(t) = 0 \text{ for } t < 0 \end{array}\right| \Leftrightarrow f(t)\,\gamma(t)$$

1) The Laplace transf. of a step function of amplitude C, that is of a ct original fction, is:

$$L\{C\gamma(t)\} = \int_{0-}^{\infty} Ce^{-st}\,dt = -\frac{C}{s}\Big|_{0-}^{\infty} = \frac{C}{s}$$

Particular case: $L\{\gamma(t)\} = \frac{1}{s} \longrightarrow L\{0\} = 0$

2) The Laplace transf. of an impulse fction of contents C

$$L\{C\delta(t)\} = \int_{0-}^{\infty} C\delta(t)\,e^{-st}\,dt = Ce^{-st}\Big|_{t=0} = C$$

3) The Laplace transf. of an exponential fction:

$$L\{e^{at}\gamma(t)\} = \int_{0-}^{\infty} e^{(a-s)t}\,dt = \frac{e^{(a-s)t}}{a-s}\Big|_{0-}^{\infty} = \frac{1}{s-a} \text{ for}$$

$Re\{s\} > Re\{a\}$

(which could also be obtained through the 5th property of the Laplace transf.)

4) By direct calc. or, even better, by the linearity of the
? Laplace transf., the Laplace images of the harmonically time-varying original fctions are obtained:

$$L\{\sin \omega t\, \gamma(t)\} = L\left\{\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\, \gamma(t)\right\} =$$

$$= \frac{1}{2j}\left(\overbrace{\frac{1}{s-j\omega}}^{s+j\omega} - \overbrace{\frac{1}{s+j\omega}}^{s-j\omega}\right) = \frac{1}{2j}\left(\frac{s+j\omega}{s^2+\omega^2} - \frac{s-j\omega}{s^2+\omega^2}\right)$$

$$= \frac{1}{\cancel{2j}} \cdot \frac{\cancel{2j}\omega}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2}$$

$$L\{\cos \omega t\, \gamma(t)\} = L\left\{\frac{e^{j\omega t} + e^{-j\omega t}}{2} \cdot \gamma(t)\right\} = \frac{1}{2}\left(\overbrace{\frac{1}{s-j\omega}}^{s+j\omega} + \overbrace{\frac{1}{s+j\omega}}^{s-j\omega}\right) =$$

$$= \frac{1}{2} \cdot \frac{2s}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}$$

5) By direct calc. or, even better, by the 3rd property of the Laplace transf., a polynomial ramp original fction has a Laplace transf. like

$$L\{C t\, \gamma(t)\} = L\left\{\int_{0-}^{t} C\gamma(\theta)\, d\theta\right\} = \frac{C}{s} L\{\gamma(t)\} = \frac{C}{s^2}$$

- in general, for $m \geq 1 \longrightarrow L\{C t^m \gamma(t)\} = C \cdot \frac{m!}{s^{m+1}}$

6) The Laplace transf. of a superior order derivative:

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = L\left\{\frac{d}{dt}\frac{df(t)}{dt}\right\} = s \cdot L\left\{\frac{df(t)}{dt}\right\} - f'(0) =$$

$$= s\left[s L\{f(t)\} - f(0)\right] - f'(0)$$

$$= s^2 L\{f(t)\} - s f(0) - f'(0)$$

- in general, for $n \geq 2$

$$L\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

$\circledast$

$$\sum_{k \in (g)} \left\{ R_k i_k + \frac{1}{sC_k} i_k + \sum_{j=1}^{L} sL_{kj} i_j + U_k - \left[E_k - \frac{u_{ck}(0)}{s} + \phi_{Lk}(0) \right] \right\} = 0$$

$$\Leftrightarrow \sum_{k \in (g)} \left(R_k i_k + \frac{1}{sC_k} i_k + \sum_{j=1}^{L} sL_{kj} i_j + U_k \right) =$$

$$= \sum_{k \in (g)} \left[E_k - \frac{u_{ck}(0)}{s} + \phi_{Lk}(0) \right]$$

- operational impedances

$$Z_{kk} = R_k + \frac{1}{sC_k} + sL_{kk} = R_k + \frac{1}{sC_k} + sL_k = Z_k$$

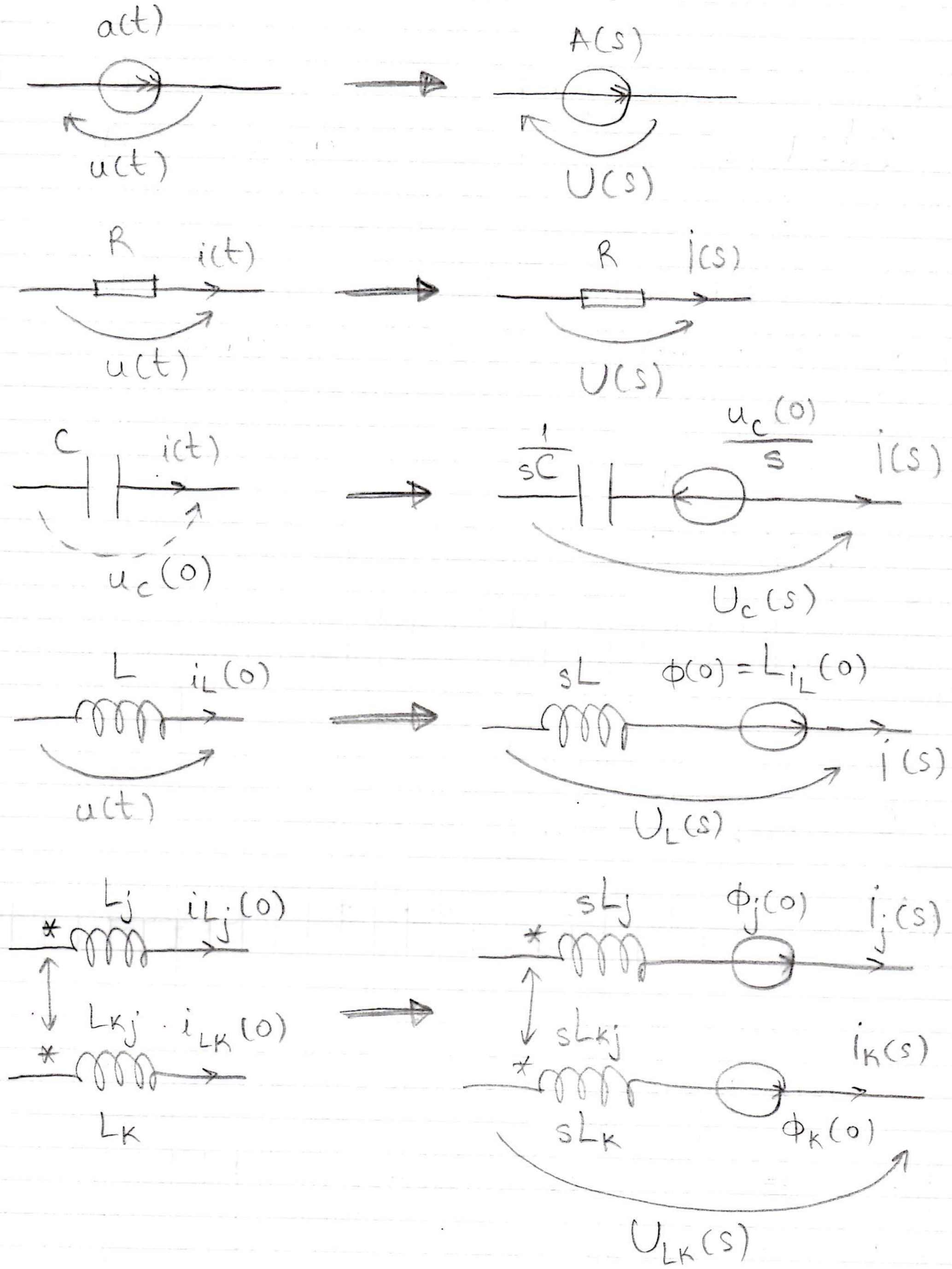
$$Z_{kj} = sL_{kj} = sL_{jk} = Z_{jk}$$

- operational Kirchhoff's voltage theorem

$$\sum_{k \in (g)} \left(Z_k i_k + \sum_{\substack{j=1 \\ j \neq k}}^{L} Z_{kj} i_j + U_k \right) = \sum_{k \in (g)} \left[E_k - \frac{u_{ck}(0)}{s} + \phi_{Lk}(0) \right]$$

Algorithm of the operational calculus of a linear circuit under time-varying conditions:

1) The operational circuit associated to the original circuit is constructed according to the correspondence rules illustrated below.

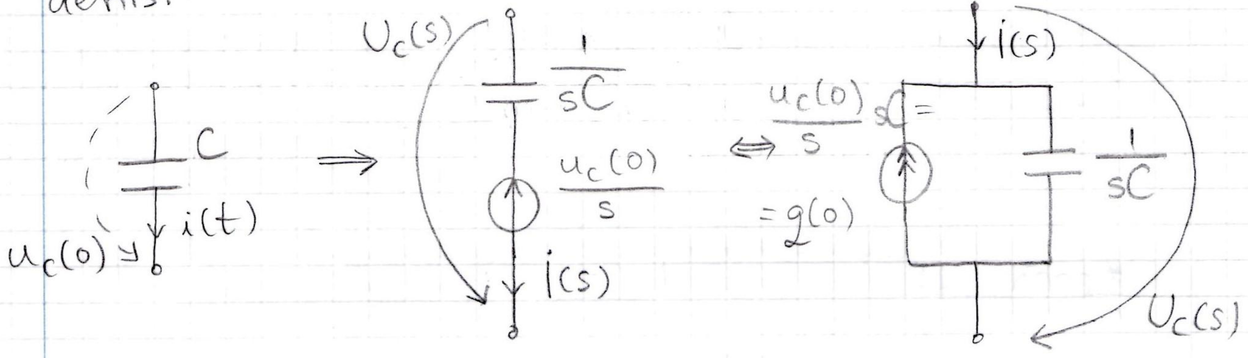


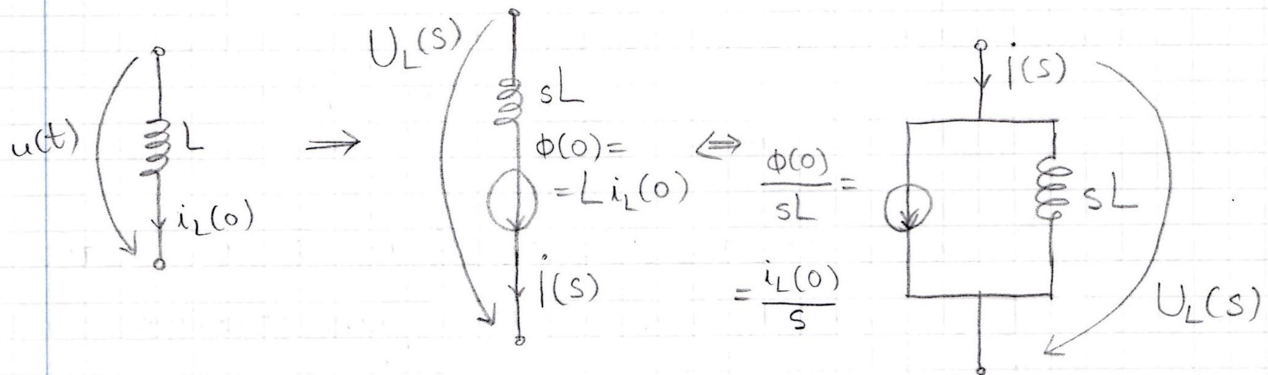
Every passive element is associated with an operational impedance and - for reactive elements - a generator equivalent with the initial conditions, and every active element is associated with an operational generator characterised by the Laplace transformation of the instantaneous characteristic of that generator.

The branches of the operational circuit carry operational currents and operational voltages are present at the terminals of operational equivalents of circuit elements.

2) The operational Kirchhoff's equations are directly formulated for the operational circuit, and their solution yields the operational unknown quantities, representing the Laplace transformations of the unknown instantaneous currents & voltages in the original circuit.

3) The time-variation of the unknown currents & voltages is obtained as the inversed Laplace transformations of their now determined operational correspondents.





Time-domain analysis of linear time-invariant circuits

Let us consider
- $x(t)$ excitation point (where signal sources are placed)
- $y(t)$ point of response (where the processed signal is analised)

- input – output relationship : $x(t) \rightarrow y(t)$

A circuit (system) is linear if

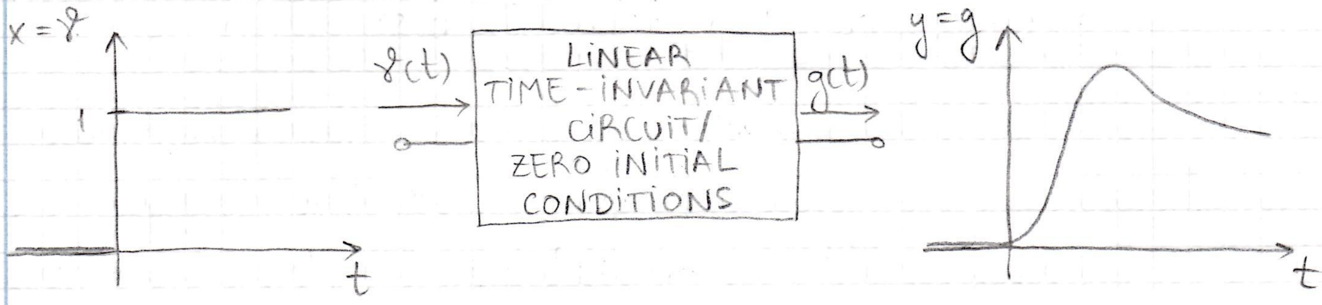
$$a \cdot x_1(t) + b \cdot x_2(t) \longrightarrow a \cdot y_1(t) + b \cdot y_2(t)$$

and it's time-invariant if

$$x(t-\tau) \longrightarrow y(t-\tau)$$

$x(t) = \gamma(t) \longrightarrow y(t) = g(t)$

$x(t) = A \cdot \gamma(t-\tau) \longrightarrow y(t) = A \cdot g(t-\tau)$ (amplified/delayed step-fction excitation)



Obs. $g(t-\tau) = 0$ for $t-\tau < 0 \Rightarrow t < \tau$

(*)

$$F(s) = \frac{N(s)}{D(s)}, \quad n = \deg(D) \geq \deg(N) = m$$

In such cases, the rational fction can be expanded as a sum of partial fractions, and, by considering the examples (3) & (5) written above, Heaviside's inversion formulae are obtained as follows:

(1a) if $D(s)$ has got n simple roots

$D(s) \equiv a_0(s-s_1)(s-s_2)\dots(s-s_n)$, then

$$L^{-1}\left\{\frac{N(s)}{D(s)}\right\} = \left[\sum_{k=1}^{n} \frac{N(s_k)}{D'(s_k)} e^{s_k t}\right] \gamma(t)$$

(1b) if $D(s)$ has got n simple roots, one of which equal to 0, $D(s) \equiv sM(s)$, $M(s) \equiv a_0(s-s_1)(s-s_2)\dots(s-s_{n-1})$, then

$$L^{-1}\left\{\frac{N(s)}{D(s)}\right\}=L^{-1}\left\{\frac{N(s)}{sM(s)}\right\}=\left[\frac{N(0)}{M(0)}+\sum_{K=1}^{m}\frac{N(s_K)}{s_K M'(s_K)}\cdot e^{s_K t}\right]\gamma(t)$$

(2) If $D(s)$ has got m multiple roots

$D(s)\equiv a_0(s-s_1)^{m_1}(s-s_2)^{m_2}\dots(s-s_p)^{m_p}$, such that $m_1+m_2+\dots+m_p=m$, then

$$L^{-1}\left\{\frac{N(s)}{D(s)}\right\}=\left[\sum_{K=1}^{p}\sum_{j=1}^{m_K}H_{Kj}\cdot t^{m_K-j}\cdot e^{s_K t}\right]\gamma(t),$$ where

$$H_{Kj}=\frac{1}{(j-1)!\,(m_K-j)!}\cdot\frac{d^{j-1}}{ds^{j-1}}\left[\frac{(s-s_K)^{m_K}\cdot N(s)}{D(s)}\right]_{s=s_K}$$

In relation with the expansion of a Laplace image as a sum of partial fractions, the last inversion formula can be completed with the observation that in the case of multiple complex roots of the denominator,

$$L^{-1}\left\{\frac{1}{[s-(\alpha+j\omega)]^m}\right\}=\frac{t^{m-1}}{(m-1)!}e^{\alpha t}e^{j\omega t}\gamma(t)$$

The operational calc. of linear circuits implies the use of direct & inversed Laplace transf. at its initial and final steps.

Let the image fctions (La place transf. of the time-varying quantities of interest) be:

$$\left.\begin{aligned}E_K&=E_K(s)=L\{e_K(t)\}=L\{e_K\}\\A_K&=A_K(s)=L\{a_K(t)\}=L\{a_K\}\end{aligned}\right\}\text{given}$$

$$\left.\begin{aligned}I_K&=I_K(s)=L\{i_K(t)\}=L\{i_K\}\\U_K&=U_K(s)=L\{u_K(t)\}=L\{u_K\}\end{aligned}\right\}\text{to be det.}$$

The linearity of the Laplace transf. allows the translation of Kirchhoff's eq. as such:

$$L\left\{\sum_{K\in(a)} i_K\right\} = L\{0\} \Leftrightarrow \sum_{K\in(a)} L\{i_K\} = 0 \Leftrightarrow \sum_{K\in(a)} I_K = 0$$

$$L\left\{\sum_{K\in(g)} u_{bK}\right\} = L\{0\} \Leftrightarrow \sum_{K\in(g)} L\{u_{bK}\} = 0 \Leftrightarrow \sum_{K\in(g)} U_{bK} = 0$$

along with Joubert's theorem,

$$L\{u_{bK}\} = L\{u_{RK} + u_{CK} + u_{LK} + u_K - e_K\} \Leftrightarrow$$

$$\Leftrightarrow L\{u_{bK}\} = L\{u_{RK}\} + L\{u_{CK}\} + L\{u_{LK}\} + L\{u_K\} - L\{e_K\}$$

$$\Leftrightarrow U_{bK} = U_{RK} + U_{CK} + U_{LK} + U_K - E_K$$

The linearity of the transf. and its action on derivatives or integrals of time-varying fctions are:

$$L\{u_{RK}\} = L\{R_K i_{RK}\} \Leftrightarrow L\{u_{RK}\} = R_K L\{i_{RK}\} \Leftrightarrow U_{RK} = R_K I_{RK}$$

↳ for the ideal resistor

$$L\{u_{CK}\} = L\left\{u_{CK}(0) + \frac{1}{C_K}\int_0^t i_{CK}\,dt\right\} \Leftrightarrow$$

$$\Leftrightarrow L\{u_{CK}\} = L\{u_{CK}(0)\} + L\left\{\frac{1}{C_K}\int_0^t i_{CK}\,dt\right\} \Leftrightarrow$$

$$\Leftrightarrow L\{u_{CK}\} = L\{u_{CK}(0)\} + \frac{1}{C_K}\cdot L\left\{\int_0^t i_{CK}\,dt\right\} \Leftrightarrow$$

$$\Leftrightarrow U_{CK} = \frac{u_{CK}(0)}{s} + \frac{1}{C_K}\cdot \underline{\qquad\qquad} \quad ?$$

↳ for the ideal capacitor with the initial voltage $u_{CK}(0)$

$$L\{u_{LK}\} = L\left\{\sum_{j=1}^{L} L_{Kj}\frac{di_{Lj}}{dt}\right\} \Leftrightarrow L\{u_{LK}\} = \sum_{j=1}^{L} L\left\{L_{Kj}\frac{di_{Lj}}{dt}\right\} \Leftrightarrow$$

$$\Leftrightarrow L\{u_{LK}\} = \sum_{j=1}^{L} L_{Kj}\, L\left\{\frac{di_{Lj}}{dt}\right\} \Leftrightarrow U_{LK} = \sum_{j=1}^{L} L_{Kj}\left[s I_{Lj} - i_{Lj}(0)\right] \Leftrightarrow$$

$$\Leftrightarrow U_{LK} = \sum_{j=1}^{L} sL_{Kj} I_{Lj} - \sum_{j=1}^{L} L_{Kj}\, i_{Lj}(0) \Leftrightarrow U_{LK} = \sum_{j=1}^{L} sL_{Kj} I_{Lj} - \Phi \ ?\underline{\qquad\qquad}$$

→ for the ideal coil in a system of coupled coils with an initial magnetic flux $\phi_{LK}($?

The substitution of these explicit operational voltages into Joubert's theorem

$$U_{bK} = R_K i_K + \frac{u_{cK}(0)}{s} + \frac{1}{sC_K} i_K + \sum_{j=1}^{L} sL_{Kj} i_j - \phi_{LK}(0) + U_K - E_K =$$

$$= R_K i_K + \frac{1}{sC_K} i_K + \sum_{j=1}^{L} sL_{Kj} i_j + U_K - \left[E_K - \frac{u_{cK}(0)}{s} + \phi_{LK}(0)\right]$$

11th Jan 2017

Laplace method

Laplace transforms for the main time fctions f(t)

$f(t)$	$F(s)$
C	$\frac{C}{s}$
t	$\frac{1}{s^2}$
$e^{\pm\alpha t}$	$\frac{1}{s \pm \alpha}$
$t \cdot e^{\pm\alpha t}$	$\frac{1}{(s\pm\alpha)^2}$
$\frac{1}{\alpha}(1-e^{-\alpha t})$	$\frac{1}{s} \cdot \frac{1}{s+\alpha}$
$\sin\alpha$	$\frac{\alpha}{s^2+\alpha^2}$
$\cos\alpha t$	$\frac{s}{s^2+\alpha^2}$
$sh\,\alpha t$	$\frac{\alpha}{s^2-\alpha^2}$
$ch\,\alpha t$	$\frac{s}{s^2-\alpha^2}$