Periodic Operation of Linear Circuits (Non-Sinusoidal)

A linear circuit operates under periodic state conditions if all electromagnetic quantities associated with its operation present a periodic time-variation of the same frequency or, equivalently, with the same period T.

$$\boxed{f(t) = f(t+mT)}$$

The study of the periodic operation of linear circuits can be done by invoking the superposition theorem: periodic time-varying functions are approached as combinations appropriately chosen simpler periodic functions of time. And the immediate candidates simpler periodic functions are the harmonic time-varying functions

$\boxed{\omega = \frac{2\pi}{T}}$ → fundamental frequency

It can be proved that a periodic time-varying function, which is sequentialy (quadratically) integrable and sectionally smooth over a period can be expressed as an linear combination of the above specied basis functions. An equivalent statement is valid under somewhat different conditions, namely for periodic time varying functions which are absolute integrable and sectionally smooth over a period.

A periodic time-varying function of period T which satisfies the above enumerated conditions admits the Fourier series expansion.

$$f(t) = \frac{1}{2}(a_0 + b_0) + \sum_{h=1}^{\infty}(a_h \cos h\omega t + b_h \sin h\omega t)$$

$$a_h = \frac{2}{T}\int_\theta^{\theta+T} f(t)\cos h\omega t\,dt \qquad b_h = \frac{2}{T}\int_\theta^{\theta+T} f(t)\cos h\omega t\,dt \quad h\in$$

$$\frac{a_0}{2} = \frac{1}{T}\int_\theta^{\theta+T} f(t)\cos\theta\omega t\,dt = \frac{1}{T}\int_\theta^{\theta+T} f(t)\,dt$$

$$\frac{b_0}{2} = \frac{1}{T}\int_\theta^{\theta+T} f(t)\sin\theta\omega t\,dt = 0$$

$$f(t) = F_0 + \sum_{h=1}^{\infty} F_h\sqrt{2}\sin(h\omega t + \varphi_h) = \sum_{h=0}^{\infty} f_h(t)$$

$$F_0 = \frac{a_0}{2} = \frac{1}{T}\int_\theta^{\theta+T} f(t)\,dt$$

$$F_h = \sqrt{\frac{a_h^2 + b_h^2}{2}}$$

In particular, the average value $f_0 = F_0$ might be named the harmonic of order zero

The harmonic components are linearly independent

The set $\{f_h\}_h = 0, 1, 2$ of harmonic components associated with fund. frequency ω are orthogonal to each other with respect to the so called scalar product

$$\langle f, g\rangle = \frac{1}{T}\int_\theta^{\theta+T} fg\,dt = \overline{fg}$$

Harmonic analysis of the periodic operation

The periodic operation of a linear el. circuit is described by the set of Kirchoff's equations.

$$\begin{cases} i_k(t) = \sum_{n=1}^{\infty} i_k^n(t) \quad ; \quad u_{tk}(t) = \sum_{n=1}^{\infty} u_{tk}^n(t) \\ \sum_{k\in(a)} i_k(t) = 0 \quad a = \overline{1, N-1} \quad ; \quad \sum_{k\in(p)} u_{tk}(t) = 0 \quad p = \overline{1, B-N+1} \end{cases}$$

Kirchoff's equations have to be satisfied separately for each harmonic component.

Kirchoff's voltage theorem is to be completed with Joubert's theorem, for each harmonic component:

$$n=0 \quad U_{tk}^0 = \begin{cases} -U_k^0 - E_k^0 & \text{for a branch including a capacitor} \\ R_k I_k^0 - U_k^0 - E_k^0 & \text{for a branch without a capacitor} \end{cases}$$

$$U_{tk}^n = \sum_{k=1}^{L} Z_{ks}^n I_s^n - U_k^n - E_k^n \quad , \quad n = 1, 2, \dots$$

$\boxed{\text{LINEAR CIRCUIT IN PERIODIC STEADY-STATE}} = \boxed{\text{LINEAR D.C. CIRCUIT } (\omega = 0)} + \boxed{\text{LINEAR A.C. CIRCUIT (freq. } \omega)} + \boxed{\text{LINEAR A.C. CIRCUIT freq } 2\omega} + \dots + \boxed{\text{LINEAR A.C. CIRCUIT (freq } n\omega)}$

The algorithm for calculation is:

1°. The harmonic Fourier series expansions are computed

2°. The circuit is studied separately for the direct current component $n=0$, and for successive harmonic components $n=1,2,\dots$ to find the harmonic components of the unknown currents and voltages.

3°. Each unknown current & voltage is sintesised as the sum of its computed harmonic components

The received active power : $P = \overline{ui} = \frac{1}{T}\int_{\theta}^{\theta+T} u(t)\, i(t)\, dt$

$$P = \overline{u \cdot i} = \left(U_0 + \sum_{k=1}^{\infty} u_k\right)\left(I_0 + \sum_{k=1}^{\infty} i_n\right)$$

$$P = U_0 I_0 + \sum_{k=1}^{\infty} U_K I_K \cos \varphi_k$$

Reactive power $\boxed{Q = \sum_{k=1}^{\infty} U_K I_K \sin \varphi_k}$

The effective value:

$$F = \sqrt{\overline{f^2}} = \sqrt{\frac{1}{T}\int_{\theta}^{\theta+T} f^2(t)\, dt} \longrightarrow \boxed{F = \sqrt{\sum_{k=0}^{\infty} F_k^2}}$$

The apparent power : $\boxed{S = U \cdot I}$

$\boxed{P^2 + Q^2 = S^2}$ is no longer verified in the case of periodic operating conditions. Consequently, a so called distortion power can be defined as:

$$\boxed{D = \sqrt{S^2 - (P^2 + Q^2)}}$$

The distorsion residual

$$F_d = \sqrt{\sum_{h=1}^{\infty} F_h^2} = \sqrt{F^2 - F_0^2 - F_1^2}$$

$$\boxed{F_d = \sqrt{\sum_{h=1}^{\infty} F_k^2} = \sqrt{F^2 - F_1^2}}$$

The distorsion coefficient: $\boxed{h_d = \frac{F_d}{F}}$