The Lorentz Forces on an Electrically Conducting Sphere in an Alternating Magnetic Field

Udaya B. Sathuvalli, and Yildiz Bayazitoglu

Abstract—**A method to calculate the Lorentz force on an electrically conducting sphere placed in an arbitrary sinusoidally varying magnetic field is developed. The crux of this method lies in expressing the external magnetic field and the eddy current density in the sphere in terms of a "source function" of the current sources and a "skin depth dependent function." The general formula for the Lorentz force is used to derive the special case of a sphere in an axisymmetric stack of circular current loops. Numerical results for this case are presented as a function of the stack geometry. Approximations of the skin depth functions for practical situations are presented. Finally, a procedure to determine the magnetic pressure distribution on the surface of a levitated liquid metal droplet is given.**

I. Introduction

THE solution to the problem of a metallic sphere placed in a magnetic field finds application in many areas of containerless processing. For example, in electromagnetic levitation melting, a small coil that is wound over the length of a few centimeters and carries a high-frequency alternating current is used to levitate and melt small metallic spheres [1], [2]. Bayazitoglu and Cerny [3] propose and study a method to produce fine metallic powders by allowing a spray of liquid metal drops to fall through the high-frequency alternating magnetic field of a long solenoid. In these and several other applications such as gradiometry [4], determination of surface tension and viscosity of liquid metals [5], the calculation of the force on an electrically conducting sphere and the heating in it are very important.

Since the time when electromagnetic levitation melting was first demonstrated experimentally [1], it has been extensively used for the measurement of thermophysical properties such as thermal diffusivity [6], surface tension [5], [7], and viscosity of high temperature liquid metals and alloys in a containerless manner. For example, when a drop of liquid metal is levitated it executes shape oscillations [5], [7], [8]. The dynamics of these shape oscillations are determined by the balance between the hydrostatic, viscous, surface tension, and electromagnetic forces. In this case, it is important to calculate the magnetic pressure on the surface of the conducting droplet of liquid metal beforehand [5], [7].

All the works that address the problem of an electrically conducting sphere in an alternating magnetic field can be classified as belonging to either the homogeneous [1], [2], [9], or the nonhomogeneous model. The homogeneous model assumes that the conducting sphere is placed in a uniform and unidirectional sinusoidally alternating magnetic field and shows that the Lorentz force on the sphere is proportional to the product of the field and its gradient [2] [as given by (28)]. This assumption gives rise to several problems. For example, the magnetic field can hardly be regarded as being uniform or unidirectional over the diameter of the sphere. Calculations of the net power absorbed by a sphere in a magnetic field by using the homogeneous model show that it underestimates the power by as much as 30% in typical laboratory-type levitation situations [10]. On the other hand, works that account for the nonuniformity of the field are highly geometry specific [11]–[13]. They are applicable only to axisymmetric systems. Often the magnetic fields produced by the laboratory coils are not quite axisymmetric [14]. Recently, Lohofer [15], [16] analyzed the problem of a conducting sphere in a magnetic field placed in an arbitrary (nonuniform and not necessarily unidirectional) sinusoidally alternating magnetic field and obtained expressions for the net power generated in the sphere, and the Lorentz force and torque on it in terms of certain functions of the "current sources" that create the external magnetic field. In this work, this model is known as the nonhomogeneous model.

In the present work, the density of eddy currents that are generated in a diamagnetic sphere placed in a sinusoidally alternating magnetic field as given in [15] is first written. Then, by using multipole expansion, the vector potential of the external magnetic field is expressed in terms of the above-mentioned "source functions." The external magnetic field is calculated by using a gradient formula. The expression for the instantaneous Lorentz force per unit volume is subsequently integrated over the volume of the sphere to obtain the net time averaged force on the sphere in terms of these source functions and a "skin depth function." Approximations for the skin depth function for use in practical situations are presented. The general formula for the net Lorentz force is then used to

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derive the force on a sphere in an axisymmetric magnetic field that is created by a stack of circular coaxial loops and numerical results in terms of the stack geometry are presented. The homogeneous model is shown to be a special case of the nonhomogeneous model, and the difference in the predictions of the Lorentz force according to the two models is estimated. Finally, a method to find the magnetic pressure distribution on the surface of a levitating liquid metal droplet by using this method is presented.

II. Analysis

Consider a diamagnetic sphere of radius R_s, electrical conductivity σ_s, and magnetic permeability μ_o, that is placed in a time varying magnetic field $\boldsymbol{B}_e(\boldsymbol{r}, t)$ (see Fig. 1). If the external magnetic field is created by a set of N current sources whose current densities are $\boldsymbol{J}_n(\boldsymbol{r}) \cos(\omega_n t)$, then its vector potential $\boldsymbol{A}_e(\boldsymbol{r}, t)$ may described by

$$\boldsymbol{A}_e(\boldsymbol{r}, t) = \sum_{n=1}^{N} \boldsymbol{A}_n(\boldsymbol{r}) \cos(\omega_n t) \tag{1a}$$

where

$$\boldsymbol{B}_e(\boldsymbol{r}, t) = \nabla \times \boldsymbol{A}_e(\boldsymbol{r}, t) \tag{1b}$$

$$\nabla \cdot \boldsymbol{A}_e(\boldsymbol{r}, t) = 0. \tag{1c}$$

For this situation, the density of the induced eddy currents is given by [15]

$$\begin{aligned} \boldsymbol{J}_s(r, u, \phi, t) &= \frac{2}{R_s^{3/2}} \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{k=1}^{\infty} \\ &\cdot \frac{\boldsymbol{I}_{n,l,m}}{J_{l+1/2}(x_{l+1/2,k})} \frac{2q_n^2}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}} \\ &\cdot r^{-1/2} J_{l+1/2}\left(x_{l+1/2,k} \frac{r}{R_s}\right) Y_l^m(u, \phi) \\ &\cdot \cos(\omega_n t + \psi_{n,k,l}), \qquad r \le R_s, \end{aligned} \tag{2a}$$

where

$$\begin{aligned} \boldsymbol{I}_{n,l,m} = R_s^l \int_{R_s}^{\infty} \int_{-1}^{1} \int_{o}^{2\pi} \\ \cdot \boldsymbol{J}_n(r, u, \phi) r^{-l+1} Y_l^{m*}(u, \phi)\, d\phi\, du\, dr \end{aligned} \tag{2b}$$

$$q_n = \frac{R_s}{\delta_n} \tag{2c}$$

$$\delta_n = \sqrt{\frac{2}{\omega_n \mu_o \sigma_s}} \tag{2d}$$

$$\varphi_{n,k,l} = \cos^{-1}\left(-\frac{2q_n^2}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}}\right) \tag{2e}$$

and $Y_l^m(u, \phi)$ are complex spherical harmonics defined by

$$Y_l^m(u, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m(u) e^{im\phi} \tag{3}$$

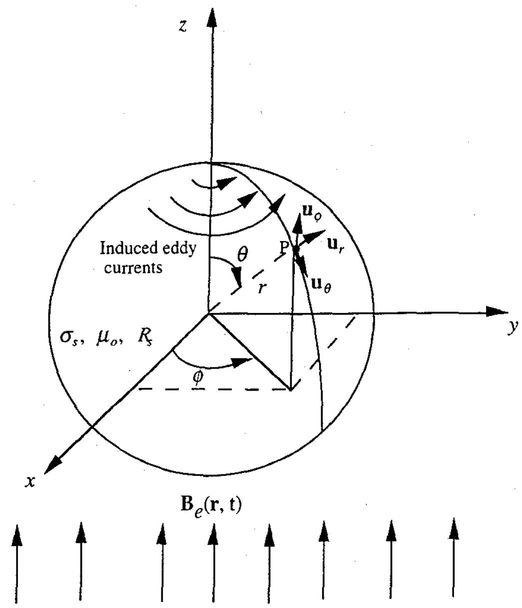

Fig. 1. A sphere placed in a magnetic field.

and the asterisk denotes complex conjugation. Also, $u = \cos\theta$ and $P_l^m(u)$ are associated Legendre polynomials of the first kind. $J_{l+1/2}(z)$ is a fractional order Bessel function of the first kind, and $x_{l+1/2,k}$ is the kth real root of

$$J_{l-1/2}(x) = 0. \tag{4}$$

The k-dependent terms in (2a) may be summed to yield

$$\begin{aligned} \boldsymbol{J}_s(r, u, \phi, t) = \frac{-4}{R_s^{3/2}} \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} q_n^2 \boldsymbol{I}_{n,l,m} r^{-1/2} \\ \cdot \operatorname{Re}\left[\Psi_l(z, r/R_s) e^{i(\omega_n t + \pi/2)}\right] Y_l^m(u, \phi) \end{aligned} \tag{5a}$$

where for real values of s

$$\Psi_l(z, s) = \frac{1}{2z} \frac{I_{l+1/2}(zs)}{I_{l-1/2}(z)} \tag{5b}$$

$$z = (1 + i) q_n \tag{5c}$$

and $I_{l\pm 1/2}(z)$ denote modified Bessel functions of the first kind. Equations (5a) and (5b) are proved in Appendix D. The above equations are valid in a spherical coordinate system whose origin is at the center of the sphere. Consequently, the current densities of the external sources that create the magnetic field must be described as seen from this coordinate system. Equation (2b) defines the complex vector $\boldsymbol{I}_{n,l,m}$ which is purely a function of the external currents that produce the magnetic field and is referred to as the "source function." Note that in (2), δ_n is the skin depth and q_n is the dimensionless ratio of the sphere radius to its skin depth at a given frequency. Finally, we must point out that the form of the induced eddy current density as suggested by (5) is particularly useful in the calculation of the magnetic pressure that acts on the surface of a levitating drop of liquid metal (see Section VI).

The net time averaged Lorentz force acting on the sphere is

$$\boldsymbol{F}_s = \underset{T\to\infty}{Lt}\ \frac{1}{T}\int_0^T \int_0^{R_s} \int_{-1}^{+1} \int_0^{2\pi} \mathrm{Re}[\boldsymbol{J}_s(\boldsymbol{r}, t)] \times \mathrm{Re}[\boldsymbol{B}_e(\boldsymbol{r}, t)]\ r^2\, d\phi\, du\, dr\, dt. \tag{6}$$

The form of (6) suggests that it is convenient to express the external magnetic field in terms of the source function $\boldsymbol{I}_{n,l,m}$, since the induced eddy current density inside the sphere has already been expressed in terms of it [according to (2)].

The vector potential of the external magnetic field can be expressed in terms of the external current densities by [17]

$$\boldsymbol{A}_n(\boldsymbol{r}) = \frac{\mu_o}{4\pi}\iiint \frac{\boldsymbol{J}_n(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\, d^3r'. \tag{7}$$

Since (7) represents the vector potential due to the external currents, the region of integration is all space that is outside the sphere. The denominator of the integrand in (7) can be written as a multipole expansion in spherical harmonics [18]

$$|\boldsymbol{r}(r, u, \phi) - \boldsymbol{r}'(r', u', \phi')|^{-1} = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\, Y_l^{m*}(u, \phi)\, Y_l^m(u', \phi') \cdot \begin{cases} \dfrac{r^l}{r'^{l+1}}, & r < r' \\ \dfrac{r'^l}{r^{l+1}}, & r > r'. \end{cases} \tag{8}$$

Substituting (8) in (7) and considering only the case $r < r'$ yields

$$\boldsymbol{A}_n(r, u, \phi) = \mu_o \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\, Y_l^{m*}(u, \phi) r^l \cdot \left[\int_{R_s^+}^{\infty}\int_{-1}^{+1}\int_0^{2\pi} \boldsymbol{J}_n(r', u', \phi') r'^{-l+1} \cdot Y_l^m(u', \phi')\, d\phi'\, du'\, dr'\right].$$

Assuming that $\boldsymbol{J}_n(r, u, \phi)$ is real for all n, the above equation in conjunction with (2b) becomes

$$\boldsymbol{A}_n(r, u, \phi) = \mu_o \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{R_s^{-l}}{2l+1}\, r^l Y_l^{m*}(u, \phi) \boldsymbol{I}_{n,l,m}^*. \tag{9a}$$

The vector potential of the external magnetic field is then given by

$$\boldsymbol{A}_e(r, u, \phi, t) = \mu_o \sum_{n=1}^{N}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{R_s^{-\ell}}{2l+1}\, r^l Y_l^{m*}(u, \phi) \boldsymbol{I}_{n,l,m}^* \cos(\omega_n t). \tag{9b}$$

The magnetic flux density of the external field can then be found by (1b). The calculation of the external magnetic field [according to (1b) and (9b)] requires the use of the general gradient formula for functions of the type $f(r)Y_l^m(u, \phi)$ [19]. However, this involves the calculation of Clebsch–Gordan coefficients which are used in the description of angular momenta in quantum mechanics. For the purposes of the present calculation, it is advantageous (from a purely algebraic point of view) to rewrite the standard spherical harmonics $Y_l^m(u, \phi)$ in a slightly modified form. Following the notation used in [20], we let

$$Y_{l,m}^e(u, \phi) = P_l^m(u) \cos m\phi \tag{10a}$$

$$Y_{l,m}^0(u, \phi) = P_l^m(u) \sin m\phi \tag{10b}$$

so that $Y_{l,m}^e$ and $Y_{l,m}^0$ are the real and imaginary parts of the standard spherical harmonic $Y_l^m(u, \phi)$, to a multiplicative factor.

Now let the complex vector (''source function'')

$$\boldsymbol{I}_{n,l,m} = \boldsymbol{U}_{n,l,m} + i\boldsymbol{V}_{n,l,m} \tag{11}$$

where both $\boldsymbol{U}_{n,l,m}$ and $\boldsymbol{V}_{n,l,m}$ are real. Then (2a) and (11) yield

$$\mathrm{Re}[\boldsymbol{J}_s(r, u, \phi, t)] = \sum_{n=1}^{N}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{k=1}^{\infty} [f_{l,k,m}^e(r, u, \phi)\boldsymbol{M}_{n,k,l,m} - f_{l,k,m}^0(r, u, \phi)\boldsymbol{N}_{n,k,l,m}] \cos(\omega_n t + \psi_{n,k,l}) \tag{12a}$$

where

$$f_{l,k,m}^{e\text{ or }0}(r, u, \phi) = r^{-1/2} J_{l+1/2}\left(x_{l+1/2,k}\,\frac{r}{R_s}\right) Y_{l,m}^{e\text{ or }0}(u, \phi) \tag{12b}$$

$$\boldsymbol{M}_{n,k,l,m} = E_{n,k,l,m}\boldsymbol{U}_{n,l,m} = E_{n,k,l,m}\,\mathrm{Re}[\boldsymbol{I}_{n,l,m}] \tag{12c}$$

$$\boldsymbol{N}_{n,k,l,m} = E_{n,k,l,m}\boldsymbol{V}_{n,l,m} = E_{n,k,l,m}\,\mathrm{Im}[\boldsymbol{I}_{n,l,m}] \tag{12d}$$

and

$$E_{n,k,l,m} = (-1)^m \frac{2}{R_s^{3/2}} \frac{1}{J_{l+1/2}(x_{l+1/2,k})} \frac{2q_n^2}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}} \cdot \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}. \tag{12e}$$

Equation (12) ensures that the induced current density is expressed solely in terms of known real functions of the external currents. It now remains to express the external magnetic field similarly.

Substitution of (11) in (9b) results in

$$\begin{aligned}\mathrm{Re}[\boldsymbol{A}_e(r, u, \phi, t)] &= \mu_o \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [(r^l Y^e_{l,m}(u, \phi))\boldsymbol{S}_{n,l,m} \\ &\quad - (r^l Y^0_{l,m}(u, \phi))\boldsymbol{T}_{n,l,m}] \cos(\omega_n t)\end{aligned} \tag{13a}$$

where

$$\begin{aligned}\boldsymbol{S}_{n,l,m} &= D_{l,m}\boldsymbol{U}_{n,l,m} \\ &= D_{l,m}\,\mathrm{Re}[\boldsymbol{I}_{n,l,m}],\end{aligned} \tag{13b}$$

$$\begin{aligned}\boldsymbol{T}_{n,l,m} &= D_{l,m}\boldsymbol{V}_{n,l,m} \\ &= D_{l,m}\,\mathrm{Im}[\boldsymbol{I}_{n,l,m}],\end{aligned} \tag{13c}$$

and

$$D_{l,m} = (-1)^m \frac{R_s^{-l}}{2l+1}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}. \tag{13d}$$

Since $\nabla \times \boldsymbol{I}_{n,l,m} = 0$, the definition of the magnetic vector potential yields,

$$\begin{aligned}\mathrm{Re}[\boldsymbol{B}_e(r, u, \phi, t)] &= \mu_o \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [\nabla(r^l Y^e_{l,m}(u, \phi)) \\ &\quad \times \boldsymbol{S}_{n,l,m} - \nabla(r^l Y^0_{l,m}(u, \phi)) \times \boldsymbol{T}_{n,l,m}] \cos(\omega_n t).\end{aligned} \tag{14}$$

It is easy to check that (14) satisfies the zero divergence condition

$$\nabla \cdot \boldsymbol{B}_e(r, u, \phi, t) = 0$$

as required by the Maxwell equations.

III. The Lorentz Force on the Sphere

The Lorentz force on the sphere can now be found from (6), (12), and (14) by direct substitution of the expressions for the current density and the external magnetic field. This substitution yields

$$\begin{aligned}\boldsymbol{F}_s = \mu_o \sum_{n,l,m} \sum_{n',l',m'} \sum_{k=1}^{\infty} \int_0^{R_s}\int_{-1}^{+1}\int_0^{2\pi} &\cdot \begin{bmatrix} \{\boldsymbol{M}_{n,k,l,m} f^e_{l,k,m}(r, u, \phi) - \boldsymbol{N}_{n,k,l,m} f^0_{l,k,m}(r, u, \phi)\} \\ \times \{\nabla(r^{l'} Y^e_{l',m'}(u, \phi)) \times \boldsymbol{S}_{n',l',m'} - \nabla(r^{l'} Y^0_{l',m'}(u, \phi)) \times \boldsymbol{T}_{n',l',m'}\} \end{bmatrix} r^2\, d\phi\, du\, dr \\ &\cdot \left[\underset{T\to\infty}{Lt} \frac{1}{T}\int_0^T \cos(\omega_n t + \psi_{n,k,l}) \cos(\omega_{n'} t)\, dt \right]\end{aligned} \tag{15}$$

where

$$\sum_{n,l,m} \to \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l}.$$

Since the fields are all harmonic, the time dependent integral in (15) reduces to

$$\begin{aligned}\frac{\omega_n}{2\pi}\int_0^{2\pi} \cos(\omega_n t + \psi_{n,k,l}) \cos(\omega_{n'} t)\, dt \\ = \frac{1}{2}\cos \psi_{n,k,l}\delta_{\omega_n,\omega_{n'}}\end{aligned} \tag{16}$$

where $\delta_{i,j}$ is the Kronecker delta function. The cross product in the volume integral can be expanded to yield the following terms:

$$\begin{aligned}X_{\mathrm{I}} = \sum_{n,l,m} \sum_{n',l',m'} \sum_{k=1}^{\infty} \cos \psi_{n,k,l}\delta_{\omega_n,\omega_{n'}} \int_0^{R_s}\int_{-1}^{+1}\int_0^{2\pi} &\cdot \boldsymbol{M}_{n,k,l,m} f^e_{l,k,m}(r, u, \phi) \\ &\times [\nabla(r^{l'} Y^e_{l',m'}(u, \phi)) \times \boldsymbol{S}_{n',l',m'}]\, r^2\, d\phi\, du\, dr\end{aligned} \tag{17a}$$

$$\begin{aligned}X_{\mathrm{II}} = -\sum_{n,l,m} \sum_{n',l',m'} \sum_{k=1}^{\infty} \cos \psi_{n,k,l}\delta_{\omega_n,\omega_{n'}} \int_0^{R_s}\int_{-1}^{+1}\int_0^{2\pi} &\cdot \boldsymbol{M}_{n,k,l,m} f^e_{l,k,m}(r, u, \phi) \\ &\times [\nabla(r^{l'} Y^0_{l',m'}(u, \phi)) \times \boldsymbol{T}_{n',l',m'}]\, r^2\, d\phi\, du\, dr\end{aligned} \tag{17b}$$

$$\begin{aligned}X_{\mathrm{III}} = -\sum_{n,l,m} \sum_{n',l',m'} \sum_{k=1}^{\infty} \cos \psi_{n,k,l}\delta_{\omega_n,\omega_{n'}} \int_0^{R_s}\int_{-1}^{+1}\int_0^{2\pi} &\cdot \boldsymbol{N}_{n,k,l,m} f^0_{l,k,m}(r, u, \phi) \\ &\times [\nabla(r^{l'} Y^e_{l',m'}(u, \phi)) \times \boldsymbol{S}_{n',l',m'}]\, r^2\, d\phi\, du\, dr\end{aligned} \tag{17c}$$

$$\begin{aligned}X_{\mathrm{IV}} = \sum_{n,l,m} \sum_{n',l',m'} \sum_{k=1}^{\infty} \cos \psi_{n,k,l}\delta_{\omega_n,\omega_{n'}} \int_0^{R_s}\int_{-1}^{+1}\int_0^{2\pi} &\cdot \boldsymbol{N}_{n,k,l,m} f^0_{l,k,m}(r, u, \phi) \\ &\times [\nabla(r^{l'} Y^0_{l',m'}(u, \phi)) \times \boldsymbol{T}_{n',l',m'}]\, r^2\, d\phi\, du\, dr\end{aligned} \tag{17d}$$

so that the net time averaged Lorentz force on the sphere is

$$F_s = \tfrac{1}{2}\mu_o[X_{\rm I} + X_{\rm II} + X_{\rm III} + X_{\rm IV}]. \tag{18}$$

It now remains to evaluate each of the terms in (18), which in turn involves the calculation of the sums of the integrals in (17). Fortunately, the four vector integrals in (17) share a common integral. The calculation of a typical term in (17) is shown in Appendix A. The evaluation of the integrals in (17) calls for the calculation of the gradient of functions of the type $r^l Y_{l,m}^{e \text{ or } 0}(u, \phi)$. These are evaluated in Appendix B.

The final expression for the net time averaged Lorentz force on the sphere is found by substituting the expressions for the terms in (18) from (A10), (A11), (A12), and (A13) of Appendix A. Thus,

$$F_s = \frac{1}{2}\mu_o \sum_{n=1}^{N}\sum_{n'=1}^{N}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2} g_l(q_n)\sqrt{\frac{2l+1}{2l+3}}\,\delta_{\omega_n,\omega_{n'}} \cdot \begin{bmatrix} (1+\delta_{m,0})U_{n,l,m} \times (u_x \times F^1_{n',l,m} + u_y \times Q^2_{n',l,m} + u_z \times K^1_{n',l,m}) \\ + (1-\delta_{m,0})V_{n,l,m} \times (u_x \times Q^1_{n',l,m} - u_y \times F^2_{n',l,m} + u_z \times K^2_{n',l,m}) \end{bmatrix} \tag{19a}$$

where the skin depth function $g_l(q_n)$ is given by

$$g_l(q_n) = \sum_{k=1}^{\infty} \frac{-1}{x^2_{l+1/2,k}} \frac{4q_n^4}{4q_n^4 + x^4_{l+1/2,k}} \tag{19b}$$

$$= \text{Re}\left[\Psi_l(z, 1) - \frac{1}{2(2l+1)}\right] \tag{19c}$$

z is given by (5c), and u_j, $j = x, y, z$ represents the unit cartesian vectors. Also,

$$F^1_{n,l,m} = \beta_1 U_{n,l+1,m-1} - \beta_2 U_{n,l+1,m+1} \tag{19d}$$

$$F^2_{n,l,m} = \beta_1 U_{n,l+1,m-1} + \beta_2 U_{n,l+1,m+1} \tag{19e}$$

$$Q^1_{n,l,m} = \beta_1 V_{n,l+1,m-1} - \beta_2 V_{n,l+1,m+1} \tag{19f}$$

$$Q^2_{n,l,m} = \beta_1 V_{n,l+1,m-1} + \beta_2 V_{n,l+1,m+1} \tag{19g}$$

$$K^1_{n,l,m} = \beta_3 U_{n,l+1,m} \tag{19h}$$

$$K^2_{n,l,m} = \beta_3 V_{n,l+1,m}. \tag{19i}$$

The β-coefficients are defined as

$$\beta_1 = \sqrt{(l-m+2)\,(l-m+1)}$$

$$\beta_2 = \sqrt{(l+m+2)\,(l+m+1)}$$

and

$$\beta_3 = 2\sqrt{(l+m+1)\,(l-m+1)}.$$

Along with (2b) and (11), (19) gives the net time averaged Lorentz force acting on a sphere placed in an alternating magnetic field. Equation (19) in conjunction with (2b) confirms that the Lorentz force is indeed proportional to the $(2l + 1)$th power of the sphere radius. Here, we must point out that Lohofer [16] presents an expression for the force on the sphere in terms of the same source functions, but a different skin depth function. The skin depth function in [16] makes use of ordinary fractional order Bessel functions as opposed to modified Bessel functions that we use in (19b).

Appendix D presents a proof of (19b). The skin depth function $g_l(q_n)$ for the Lorentz force is analogous to the function $H_l(q_n)$ which appears in the expression for the power absorbed in the sphere placed in an alternating magnetic field [15]. The function $g_l(q_n)$ depicts the frequency dependence of every mode in (19a) (see Fig. 2). Since modified Bessel functions can be expressed in terms of circular functions [21], the special case of $l = 1$ may be obtained explicitly. For $l = 1$,

$$g_1(q_n) = \frac{1}{4q_n}\frac{\sinh 2q_n - \sin 2q_n}{\cosh 2q_n - \cos 2q_n} - \frac{1}{6}. \tag{20a}$$

A comparison to the skin depth function derived by Rony [2] [see (28b)] reveals that

$$G(q_n) = -6g_1(q_n). \tag{20b}$$

This suggests that the homogeneous model corresponds to the $l = 1$ mode in the general expression for the force.

A note about the calculation of the skin depth function, $g_l(q_n)$, is appropriate here since it involves the evaluation of ratios of fractional order modified Bessel functions of complex arguments. Lentz [22] presents a method to evaluate such ratios in terms of continued fractions. Since modified Bessel functions can be eventually expressed in terms of ordinary Bessel functions, the Lentz algorithm may be used to evaluate $g_l(q_n)$ for arbitrary values of q_n. However, in most practical levitation situations, the ratio of the specimen radius to skin depth, q_n, is of the order of 40 to 50. In powder production applications, this ratio is seldom greater than 2. It is therefore worthwhile to examine the behavior of the skin depth function $g_l(q_n)$, for large and small limiting values of q_n. The high-frequency limit (or large q_n) can be shown to be (see Appendix D)

$$\underset{q\to\infty}{Lt}\,[g_l(q)] = \frac{-1}{2(2l+1)}. \tag{21a}$$

This is physically reasonable since (21a) implies that increasing the frequency indefinitely does not correspondingly increase the force (see Fig. 2). On the other hand, when q_n is small, it can be shown (see Appendix D) that

$$\underset{q_n\to 0}{Lt}\,[g_l(q_n)] = -\frac{4q_n^4}{(2l+1)^3\,(2l+3)\,(2l+5)}. \tag{21b}$$

Since, $I_{l+1/2}(z) \approx I_{l-1/2}(z)$ for large values of z, the skin depth function may be approximated by

$$g_l(q_n) \approx \frac{1}{4q_n} - \frac{1}{2(2l+1)} \tag{21c}$$

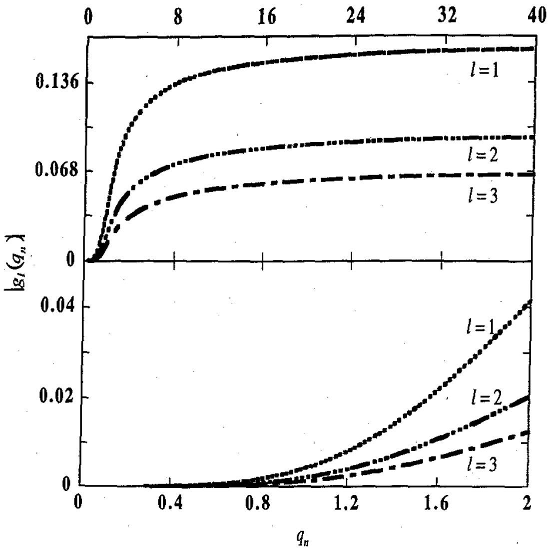

Fig. 2. The behavior of the skin depth function.

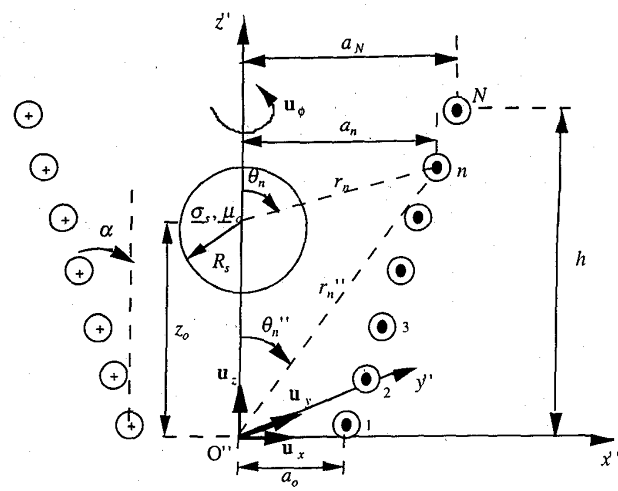

Fig. 3. A sphere on the axis of a stack of loops.

for intermediate values of $q_n > 2$. For $l = 1$, (21b) and (21c) give

$$-6 \underset{q_n \to 0}{Lt} [g_1(q_n)] = 0.025397 q_n^4 \tag{21d}$$

and

$$-6 g_1(q_n) \approx 1 - \frac{3}{2q_n} \tag{21e}$$

respectively. These are the limiting values for the skin depth function obtained by using the homogeneous model of Rony [2, (12) and (15)]. The relations in (21) obviate the need for the Lentz algorithm, which at the minimum requires the use of a computer program to evaluate the ratios of fractional order Bessel functions.

IV. The Axisymmetric Case

Fig. 3 shows a sphere placed along the axis of a conical stack of coaxial loops. The vector source function $\boldsymbol{I}_{n,l,m}$ for this arrangement can be shown to be [15], [10]

$$\boldsymbol{I}_{n,l,m} = \sqrt{\frac{\pi}{2}}\, I_{n,l}[-\delta_{m,1}(i\boldsymbol{u}_x + \boldsymbol{u}_y) + \delta_{m,-1}(-i\boldsymbol{u}_x + \boldsymbol{u}_y)] \tag{22a}$$

where

$$I_{n,l} = I_n \sqrt{[(2l+1)]/[l(l+1)]}\,(R_s/r_n)^l \sin\theta_n P_l^1(\cos\theta_n) \tag{22b}$$

and I_n is the current flowing in the nth loop. The form of (11) allows the following:

$$\boldsymbol{U}_{n,l,m} = \boldsymbol{u}_y I_{n,l} \sqrt{\frac{\pi}{2}} (-\delta_{m,1} + \delta_{m,-1}) \tag{23a}$$

$$\boldsymbol{V}_{n,l,m} = -\boldsymbol{u}_x I_{n,l} \sqrt{\frac{\pi}{2}} (\delta_{m,1} + \delta_{m,-1}). \tag{23b}$$

It is now merely a matter of substituting (23) into (19) and plodding through the various steps. Passing lightly over the finer algebraic details, the following points are noted: 1) only two terms that denote the transverse (i.e., x and y) components survive the triple vector products in (19a); 2) however, these terms do not survive the summation over the index m due to the presence of factors such as $(-\delta_{m,1} + \delta_{m,-1})$ $(\pm\delta_{m-1,1} + \delta_{m+1,-1})$; and 3) the two terms that survive the cross products and the summation over m are along the z-axis. The net Lorentz force is directed along the positive axis of the stack, and is given by

$$\boldsymbol{F}_s = \boldsymbol{u}_z \pi\mu_o \sum_{n=1}^{N} \sum_{n'=1}^{N} \sum_{l=1}^{\infty} g_l(q_n)\delta_{\omega_n,\omega_{n'}} \frac{2l+1}{l+1} I_n I_{n'} \cdot \sin\theta_n \sin\theta_{n'} P_l^1(\cos\theta_n) P_{l'}^1(\cos\theta_{n'}) \frac{R_s^{2l+1}}{r_n^l r_{n'}^{l+1}}. \tag{24}$$

Brisley and Thornton [11] obtain an identical relation, albeit in a slightly different notation, by calculating the *force exerted on the individual loops* by the eddy current field outside the sphere.

It is convenient to nondimensionalize (24) by choosing the least radius of the stack of loops as the length scale a_o ($= a_1$) and I_o as the current scale. With reference to Fig. 3, (24) becomes

$$\frac{\boldsymbol{F}_s}{\mu_o I_o^2} = \boldsymbol{u}_z \pi \sum_{n=1}^{N} \sum_{n'=1}^{N} \sum_{l=1}^{\infty} g_l(q_n) \frac{2l+1}{l+1} \bar{I}_n \bar{I}_{n'} \cdot \sin\theta_n \sin\theta_{n'} P_l^1(\cos\theta_n) P_{l'}^1(\cos\theta_{n'}) \frac{\bar{R}_s^{2l+1}}{\bar{r}_n^l \bar{r}_{n'}^{l+1}} \tag{25a}$$

where

$$\bar{r}_n = \sqrt{(\bar{z}_o - \gamma c_n)^2 + (\gamma c_n \tan\alpha + 1)^2} \quad (25b)$$

$$\tan\theta_n = \frac{\gamma c_n \tan\alpha + 1}{\bar{z}_o - \gamma c_n} \quad (25c)$$

and $\bar{I}_n = I_n/I_o$, $\bar{r}_n = r_n/r_o$, $c_n = (n-1)/(N-1)$, $\bar{z}_o = z_o/a_o$, $\gamma = h/a_o$, $\bar{R}_s = R_s/a_o$. Note that $\bar{z}_o$ is the scaled height of the center of the sphere from the center of the bottom loop in Fig. 3. For the special case of a single loop (i.e., $N = 1$), put $\gamma = 0$ and $n = 1$ in (25) to obtain

$$\frac{\boldsymbol{F}_s}{\mu_o \bar{I}_1^2} = \boldsymbol{u}_z \pi \sin^2\theta_1 \sum_{l=1}^{\infty} g_l(q) \frac{2l+1}{l+1} \cdot P_l^1(\cos\theta_1) P_{l+1}^1(\cos\theta_1) \left(\frac{\bar{R}_s}{\bar{r}_1}\right)^{2l+1}. \quad (26a)$$

Equations (25) and (26a) are reasonable, since they dictate that the force on the sphere should vanish at points far removed from the coil. Further, it is easy to check that the expression for the force in (26a) is an odd function of the position of the sphere with respect to the center of the loop, i.e.,

$$\boldsymbol{F}_s|_{\theta_1} = -\boldsymbol{F}_s|_{\pi-\theta_1}. \quad (26b)$$

From a physical standpoint, this is expected since a diamagnetic body tends to move to a region of weaker field strength.

V. The Lorentz Force According to the Homogeneous Model

The homogeneous model proceeds by assuming that the sphere is placed in a uniform and unidirectional external magnetic field. Let this magnetic field be given by $\boldsymbol{B}_e = \boldsymbol{B}_o \cos\omega t = \boldsymbol{u}_z B_o \cos\omega t$. In spherical coordinates, the vector potential of this field can be shown to be [17]

$$\boldsymbol{A}_e = \boldsymbol{u}_\phi \tfrac{1}{2} B_o r \sin\theta. \quad (27)$$

Based on this assumption, Rony [2] shows that the Lorentz force on the sphere is given by

$$\boldsymbol{F}_s = -\frac{2\pi R_s^3}{\mu_o} G(q)\, (\boldsymbol{B}_o \cdot \nabla)\boldsymbol{B}_o \quad (28a)$$

where

$$G(q) = 1 - \frac{3}{2q}\frac{\sinh(2q) + \sin(2q)}{\cosh(2q) - \cos(2q)}. \quad (28b)$$

In order to calculate the Lorentz force on the sphere for this configuration of the field by using the present method, the appropriate source function $\boldsymbol{I}_{n,l,m}$ must be found. For a uniform z-directed field, $\boldsymbol{I}_{n,l,m}$ has been shown to be [10]

$$\boldsymbol{I}_{n,l,m} = \sqrt{\frac{3\pi}{2}}\left(\frac{B_o R_s}{\mu_o}\right)\delta_{l,1}[(i\boldsymbol{u}_x + \boldsymbol{u}_y)\delta_{m,1} - (-i\boldsymbol{u}_x + \boldsymbol{u}_y)\delta_{m,-1}] \quad (29)$$

which, when coupled with (11), gives

$$\boldsymbol{U}_{n,l,m} = \boldsymbol{u}_x \sqrt{\frac{3\pi}{2}}\left(\frac{B_o R_s}{\mu_o}\right)\delta_{l,1}(\delta_{m,1} - \delta_{m,-1}) \quad (30a)$$

$$\boldsymbol{V}_{n,l,m} = \boldsymbol{u}_y \sqrt{\frac{3\pi}{2}}\left(\frac{B_o R_s}{\mu_o}\right)\delta_{l,1}(\delta_{m,1} + \delta_{m,-1}). \quad (30b)$$

These equations may be substituted in the general force equation (19). The principal steps are essentially similar to the axisymmetric case of the previous section, except that all the terms in the force expression vanish. None of them survives summation over the index l due to the presence of the factor $\delta_{l,1}\delta_{l+1,1}$. Therefore, the present method when applied to the sphere in the homogeneous unidirectional field implies that the Lorentz force is zero. Clearly, this result does not agree with (28). It must, however, be anticipated, since paradoxically enough (28) predicts that the Lorentz force vanishes in the absence of a field gradient. In other words, the homogeneous model begins by assuming a homogeneous field and derives a nonhomogeneous field as a precondition for a nonzero Lorentz force! This is not surprising since the homogeneous model calculates the force on the sphere by assuming it to be a small dipole that is placed in a nonhomogeneous field [2].[1]

In Appendix E, it is shown that for a very small sphere placed on the axis of a circular loop, the homogeneous model overestimates the Lorentz force by as much as 33%. This assertion must be tested experimentally. Fromm and Jehn [9] measure the forces on small copper spheres placed in the field of a circular loop. However, the radius of the smallest sphere that they use is only a fourth of the radius of the circular loop. The result of Appendix E is very likely valid for the limiting case of a small sphere. It suggests that the forces on smaller spheres (one-tenth of the loop radius and smaller) might be overestimated by (28). These results will be of interest to workers in the area of electromagnetic powder production, where very small droplets of liquid metal are allowed to fall through the magnetic field of a long large diameter solenoid [3].

It is worthwhile to consider the limiting case of a sphere with a very small radius, i.e., $R_s \to 0$. The source function for such a sphere placed in an arbitrary magnetic field is given by [10]

$$\boldsymbol{I}_{n,l,m} = \frac{2\sqrt{\pi}}{\mu_o} A_n(r_o, u_o, \phi_o)\delta_{l,0}\delta_{m,0} \quad (31)$$

where (r_o, u_o, ϕ_o) are the coordinates of the center of the sphere. Since the source function vanishes identically for all values of $l = m \neq 0$, (19) indicates a net zero Lorentz force on the sphere. This is an expected result since a magnetic field cannot exert a force on a zero-radius sphere. Also, in the case of an extremely small sphere, the magnetic field is essentially uniform over its volume. The sphere is not large enough to sense the inhomogene-

[1]The force on a dipole of moment $\boldsymbol{M}$ placed in a nonhomogeneous magnetic field $\boldsymbol{B}$ is $(\boldsymbol{M} \cdot \nabla)\boldsymbol{B}$.

ity of the magnetic field and hence does not experience a force.

VI. The Magnetic Pressure on the Surface of a Conducting Droplet

When the conducting sample is allowed to levitate, it melts after a short time (~ 100 s) [23]. The Lorentz force drives the liquid metal flow in the droplet. If shape oscillations are induced in the droplet, its dynamics can be analyzed to yield the surface tension of the liquid metal. For an inviscid fluid oscillating in an inviscid medium, the hydrodynamic equation for the oscillating droplet is given by [7]

$$\rho\left(\frac{\partial \boldsymbol{u}(\boldsymbol{r}, t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r}, t) \cdot \nabla\sigma(\boldsymbol{r}, t)\right) = -\nabla p(\boldsymbol{r}, t) - \rho \boldsymbol{g} + \mathrm{Re}[\boldsymbol{J}_s(\boldsymbol{r}, t)] \times \mathrm{Re}[\boldsymbol{B}_e(\boldsymbol{r}, t)] \tag{32}$$

where $\boldsymbol{u}(\boldsymbol{r}, t)$ is the bulk fluid velocity, $\sigma(\boldsymbol{r}, t) = 0$ is the equation for the surface of the droplet, $p(\boldsymbol{r}, t)$ is the pressure, ρ is the fluid density, and $\boldsymbol{g}$ is the acceleration due to gravity. However, at very high frequencies, the induced eddy currents are confined to the skin depth and the Lorentz force does not act on the bulk of the fluid [7]. Instead, it manifests itself as an effective magnetic pressure, $p_{\mathrm{mag}}(r, u, \phi, t)$, on the surface of the droplet. The details of how such an external pressure affects the dynamics of the droplets are given in [5] and [7]. For the present, we indicate how our method may be directly used to obtain the magnetic pressure distribution on the droplet surface. In the limiting case of very high frequencies (i.e., $q_n \to \infty$), the magnetic pressure decays exponentially from the surface of the droplet and can be regarded as a superficial quantity. Therefore,

$$\nabla p_{\mathrm{mag}}(r, u, \phi, t) = \underset{q_n \to \infty}{Lt} \mathrm{Re}[\boldsymbol{J}_s(r, u, \phi, t)] \times \mathrm{Re}[\boldsymbol{B}_e(r, u, \phi, t)].$$

Thus, the pressure difference $\Delta p_{\mathrm{mag}}(u, \phi, t)$ between the outside and the inside of the droplet surface is simply

$$\begin{aligned}\Delta p_{\mathrm{mag}}(u, \phi, t) &= p_{\mathrm{out,\,mag}} - p_{\mathrm{in,\,mag}} \\ &= \int_{r=0}^{R_s} \nabla p_{\mathrm{mag}}(u, \phi, t) \cdot d\boldsymbol{r} \\ &= \int_{r=0}^{R_s} \left(\underset{q_n \to \infty}{Lt} \mathrm{Re}[\boldsymbol{J}_s(r, u, \phi, t)] \times \mathrm{Re}[\boldsymbol{B}_e(r, u, \phi, t)]\right) \cdot d\boldsymbol{r}. \end{aligned} \tag{33}$$

It now remains to obtain the limiting value of the current density as $q_n \to \infty$. This is easily obtained from (5a) by using the asymptotic values for the modified Bessel functions. This process yields

$$\underset{q_n \to \infty}{Lt} [\boldsymbol{J}_s(r, u, \phi, t)] = -\frac{\sqrt{2}}{R_s} \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} q_n \boldsymbol{I}_{n,l,m} f_n(r, t) Y_l^m(u, \phi) \tag{34a}$$

where

$$f_n(r, t) = \frac{1}{r} e^{-q_n(1 - r/R_s)} \cos\left\{q_n\left(1 - \frac{r}{R_s}\right) - \omega_n t - \frac{\pi}{4}\right\}. \tag{34b}$$

Note that (34) is a physically reasonable and an expected result, since it indicates an exponentially decaying current density toward the center of the sphere. It is interesting that (34) compares well to the result of Van Bladel [24, (9.22)] for a conducting sphere placed in a uniform and unidirectional magnetic field. The magnetic pressure distribution on the surface can now be obtained by resolving the induced current density and the external magnetic field into their real and imaginary parts and then evaluating the integral in (33) by using (34).

VII. Results and Discussion

The results in the paper by Brisley and Thornton [11] provide a benchmark to check the results obtained here. An order of magnitude calculation for a 0.5-cm-radius copper sphere ($\sigma_s = 2.5 \times 10^7/\Omega - m$) placed 1 cm from the center of a loop of radius 1 cm is reported in [11]. The loop carries a current of 1320 A at 400 kHz. This current is sufficient to balance the weight of the sphere (45.72 mN). Using these values in (26a) yields a value of 46.2 mN.

Fig. 4 shows the variation of the magnitude of the Lorentz force along the axis of a single loop for three different values of the radius-to-skin depth ratio. The figure indicates that the force is a very weak function of q, for large values. Also, note that the case of $q = 31.4$ corresponds to the case shown in Fig. 4 of [11]. The difference in the scaled values along the force axis arises from the present choice of rationalized MKS units. When the values of the forces shown in Fig. 4 are multiplied by a conversion factor of 4π (to convert from rationalized MKS to electromagnetic units), the curve reported in [11] is recovered exactly. Recall that (28) predicts that the Lorentz force on a sphere varies as the product of the field and the field gradient. The magnetic field due to a circular loop has only an axial component on its axis, and is given by (E1) of Appendix E. The maximum of the product of the function in (E1) and its derivative occurs at $\bar{z}_o = 0.378$, which is close to the point at which the force shown in Fig. 4 peaks.

The top half of Fig. 5 shows the variation of the force along the axis of a right circular stack with two and five loops. The effect of the individual loops is clear. There are as many peaks as there are loops. In the case of only two loops, the loops are situated at $z_o/h = 0$ and $z_o/h = 1$. Then the field in between the two loops is relatively gradient free, and thus the force is close to zero. As in the case of the single loop, the peak occurs at a point just above the top loop. When the number of turns is increased to five (5), there is significant variation of the field over the length of the stack, and this is reflected in the force. In fact, when the number of loops per unit length is large,

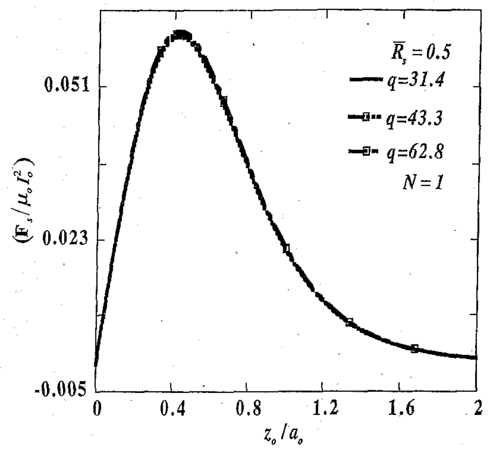

Fig. 4. The force on a sphere along the axis of a circular loop.

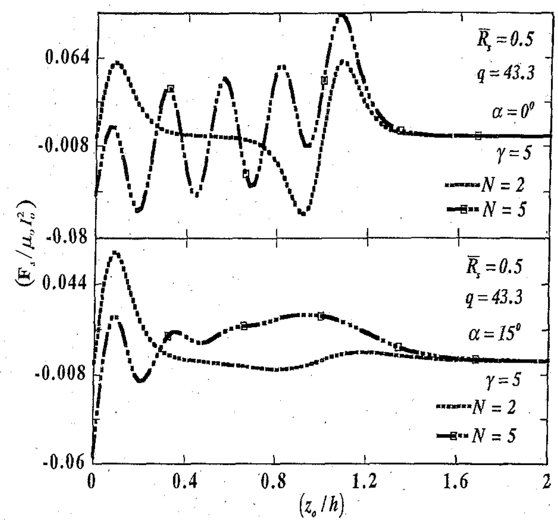

Fig. 5. Force on a sphere along the axis of a stack of loops.

the variations die out completely, and the force profile becomes smooth within the coil [14]. The bottom half of Fig. 5 shows the same results for a conical stack of 15° semiangle. Due to the increasing diameter of the loops with height, the field and the field gradient become smaller with increasing distance from the bottom of the stack. Consequently, the effect of the individual loops is not seen. The effect of a nonzero semiangle is to reduce the average magnitude of the force.

VIII. Conclusions

A method of calculating the Lorentz forces on a sphere has been presented. The method relies on expressing the induced eddy current density and the magnetic flux density of the external field in terms of certain "source functions" of the current sources that create the field. Once the "source functions" for a given configuration, such as a coil, are known, the force on the sphere may be evaluated. Further, the present method calculates the Lorentz force per unit volume by resolving the current density and the magnetic flux density of the external field into real and imaginary parts. This method has the advantage that it can be used to obtain the pressure distribution on the surface of a levitating liquid metal droplet. We conjecture that this method can also be used to find the magnetic pressure distribution on the surface of a slightly deformed aspherical liquid metal droplet in order to determine the surface tension (of a liquid metal) by studying its dynamic under normal gravity (1 g) conditions.

Acknowledgment

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Appendix A
The Evaluation of X_{I}

The substitution of the gradient formula, (B6) and (12c) and (13b) into (17a), leads to

$$X_{\mathrm{I}} = \sum_{j=x,y,z} X_{\mathrm{I},j} \tag{A1}$$

where

$$X_{\mathrm{I},j} = \sum_{n,l,m} \sum_{n',l',m'} \sum_{k=1}^{\infty} \cos \psi_{n,k,l} \delta_{\omega_n,\omega_{n'}} E_{n,k,l,m} D_{l',m'} U_{n,l,m} \times (\boldsymbol{u}_j \times \boldsymbol{U}_{n',l',m'}) \Lambda^{e,e}_{l,k,m,j} \tag{A2}$$

and

$$\Lambda^{p,q}_{l,k,m,j} = \int_0^{R_s} \int_{-1}^{+1} \int_0^{2\pi} f^p_{l,k,m}(r, u, \phi) \frac{\partial}{\partial j} \cdot \{r^{l'} Y^q_{l',m'}(u, \phi)\} r^2 \, d\phi \, du \, dr, \qquad j = x, y, z \quad p, q = e, 0. \tag{A3a}$$

The calculation of the various terms of (17a) finally involve the calculation of triple integrals of the type shown in (A3a). As an illustrative case, consider the case $j = x$, $p = q = e$. Equation (A3a) then becomes [after making use of (12b), (B3) and (B4)]

$$\begin{aligned}
\Lambda^{e,e}_{l,k,m,x} &= \int_0^{R_s} \int_{-1}^{+1} \int_0^{2\pi} r^{-1/2} J_{l+1/2}\left(x_{l+1/2,k} \frac{r}{R_s}\right) Y^e_{l,m}(u, \phi) \frac{\partial}{\partial x} (r^{l'} Y^e_{l',m'}) r^2 \, d\phi \, du \, dr \\
&= \frac{1}{2} \left[\begin{aligned} &-\left\{ \int_0^{R_s} \int_{-1}^{+1} \int_0^{2\pi} r^{l'+1/2} J_{l+1/2}\left(x_{l+1/2,k} \frac{r}{R_s}\right) Y^e_{l,m}(u, \phi) Y^e_{l'-1,m'+1} \, d\phi \, du \, dr \right\} \\ &+ \left\{ (l'+m')(l'+m'-1) \int_0^{R_s} \int_{-1}^{+1} \int_0^{2\pi} r^{l'+1/2} J_{l+1/2}\left(x_{l+1/2,k} \frac{r}{R_s}\right) Y^e_{l,m}(u, \phi) Y^e_{l'-1,m'-1} \, d\phi \, du \, dr \right\}. \end{aligned} \right]
\end{aligned} \tag{A3b}$$

The orthogonality relation (C1) implies that $l' = l + 1$, $m' = m \pm 1$ for nonzero values of the above integral. This enables the evaluation of the r-integral in both the terms above to yield

$$\int_0^{R_s} r^{l+3/2} J_{l+1/2}\left(x_{l+1/2,k}\frac{r}{R_s}\right) dr = \frac{R_s^{l+5/2}}{x^2_{l+1/2,k}}(2l+1)J_{l+1/2,k}(x_{l+1/2,k}). \tag{A4}$$

Finally,

$$\Lambda^{e,e}_{l,k,m,x} = \left[-\frac{R_s^{l+5/2}}{x^2_{l+1/2,k}} J_{l+1/2}(x_{l+1/2,k})\frac{(l+m)!}{(l-m)}\pi(1+\delta_{m,0})\delta_{l,l'-1}\{\delta_{m,m'+1} - (l+m+2)(l+m+1)\delta_{m,m'-1}\}\right]. \tag{A5}$$

For real values of ν, the following relations are useful in evaluating the r-integrals [21]:

$$\int_0^z z^{\nu+1}J_\nu(z)\,dz = z^{\nu+1}J_{\nu+1}(z) \tag{A6a}$$

$$J_{\nu+1}(z) + J_{\nu-1}(z) = \frac{2\nu}{z}J_\nu(z). \tag{A6b}$$

After some straightforward algebra, the term $\boldsymbol{X}_{\mathrm{I},x}$ can now be found from (A2), (A5), (12e), and (13d) as

$$\boldsymbol{X}_{\mathrm{I},x} = \sum_n\sum_{n'}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left\{\frac{1}{2}\sum_{k=1}^{\infty}\frac{-1}{x^2_{l+1/2,k}}\frac{4q_n^4}{(4q_n^4+x^4_{l+1/2,k})}\right\}\delta_{\omega_n,\omega_{n'}}(1+\delta_{m,0})\sqrt{\frac{2l+1}{2l+3}} \cdot\left[\boldsymbol{U}_{n,l,m}\times\left\{\boldsymbol{u}_x\times\begin{pmatrix}(l-m+2)(l-m+1)\boldsymbol{U}_{n',l+1,m-1}\\(l+m+2)(l+m+1)\boldsymbol{U}_{n',l+1,m+1}\end{pmatrix}\right\}\right]. \tag{A7}$$

In (A5), the sums over the indices l' and m' vanish due to the presence of the Kronecker delta functions $\delta_{l,l'-1}$ and $\delta_{m,m'\pm1}$. Note that the expression for $\cos\psi_{n,k,l}$ from (2e) has been used in arriving at (A7).

The other two terms in (A1) can be found by following a similar set of steps. This yields,

$$\boldsymbol{X}_{\mathrm{I},y} = 0 \tag{A8}$$

and

$$\boldsymbol{X}_{\mathrm{I},z} = \sum_n\sum_{n'}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left\{\frac{1}{2}\sum_{k=1}^{\infty}\frac{-1}{x^2_{l+1/2,k}}\frac{4q_n^4}{(4q_n^4+x^4_{l+1/2,k})}\right\}\delta_{\omega_n,\omega_{n'}}(1+\delta_{m,0})\sqrt{\frac{2l+1}{2l+3}} \cdot[\boldsymbol{U}_{n,l,m}\times\{\boldsymbol{u}_z\times(2\sqrt{(l\pm m+1)(l-m+1)}\boldsymbol{U}_{n',l+1,m})\}]. \tag{A9}$$

Finally, (A1), (A7), (A8), and (A9) result in the following expression for the first term in (18):

$$\boldsymbol{X}_{\mathrm{I}} = \sum_n\sum_{n'}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{1}{2}g_l(q_n)\delta_{\omega_n,\omega_{n'}}(1+\delta_{m,0})\sqrt{\frac{2l+1}{2l+3}} \cdot\left[\boldsymbol{U}_{n,l,m}\times\left\{\begin{matrix}\boldsymbol{u}_x\times\begin{pmatrix}\sqrt{(l-m+2)(l-m+1)}\boldsymbol{U}_{n',l+1,m-1}\\-\sqrt{(l+m+2)(l+m+1)}\boldsymbol{U}_{n',l+1,m+1}\end{pmatrix}\\+\boldsymbol{u}_z\times(2\sqrt{(l+m+1)(l-m+1)}\boldsymbol{U}_{n',l+1,m})\end{matrix}\right\}\right]. \tag{A10}$$

The remaining three terms in (18) can be obtained by a similar procedure. For the sake of brevity, only the final expression for each of these terms is given below:

$$X_{\mathrm{II}} = \sum_{n}\sum_{n'}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{1}{2}g_l(q_n)\delta_{\omega_n,\omega_{n'}}(1+\delta_{m,0})\sqrt{\frac{2l+1}{2l+3}} \cdot \left[U_{n,l,m}\times\left\{u_y\times\begin{pmatrix}\sqrt{(l-m+2)(l-m+1)}V_{n',l+1,m-1}\\ +\sqrt{(l+m+2)(l+m+1)}V_{n',l+1,m+1}\end{pmatrix}\right\}\right] \tag{A11}$$

$$X_{\mathrm{III}} = -\sum_{n}\sum_{n'}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{1}{2}g_l(q_n)\delta_{\omega_n,\omega_{n'}}(1-\delta_{m,0})\sqrt{\frac{2l+1}{2l+3}} \cdot \left[V_{n,l,m}\times\left\{u_y\times\begin{pmatrix}\sqrt{(l-m+2)(l-m+1)}U_{n',l+1,m-1}\\ +\sqrt{(l+m+2)(l+m+1)}U_{n',l+1,m+1}\end{pmatrix}\right\}\right] \tag{A12}$$

$$X_{\mathrm{IV}} = \sum_{n}\sum_{n'}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{1}{2}g_l(q_n)\delta_{\omega_n,\omega_{n'}}(1-\delta_{m,0})\sqrt{\frac{2l+1}{2l+3}} \cdot \left[V_{n,l,m}\times\left\{\begin{matrix}u_x\times\begin{pmatrix}\sqrt{(l-m+2)(l-m+1)}V_{n',l+1,m-1}\\ -\sqrt{(l+m+2)(l+m+1)}V_{n',l+1,m+1}\end{pmatrix}\\ +u_z\times\left(2\sqrt{(l+m+1)(l-m+1)}V_{n',l+1,m}\right)\end{matrix}\right\}\right]. \tag{A13}$$

Appendix B
The Calculation of the Gradient of Functions of the Type $r^l Y_{l,m}^{e\text{ or }0}(u,\phi)$

In evaluating the volume integrals in (17), it is convenient to employ a cartesian system of axes, since the unit vectors in this system are space independent throughout the region of integration (i.e., the volume of the sphere). Hence, it is easier to obtain the gradient formula in cartesian components. The definition of the gradient gives

$$\nabla(r^l Y_{l,m}^{e\text{ or }0}(u,\phi)) = \sum_{j=x,y,z} u_j\frac{\partial}{\partial j}[r^l Y_{l,m}^{e\text{ or }0}(u,\phi)] = \sum_{j=x,y,z} u_j\frac{\partial}{\partial j}\left[r^l P_l^m(u)\begin{matrix}\cos\\ \sin\end{matrix} m\phi\right]. \tag{B1}$$

$$\nabla\{r^l Y_{l,m}^{e}(u,\phi)\} = \frac{1}{2}r^{l-1}\begin{bmatrix} u_x\{-Y_{l-1,m+1}^e + (l+m)(l+m-1)Y_{l-1,m-1}^e\}\\ -u_y\{Y_{l-1,m+1}^0 + (l+m)(l+m-1)Y_{l-1,m-1}^0\}\\ +u_z\{2(l+m)Y_{l-1,m}^e\}\end{bmatrix} \tag{B6}$$

$$\nabla\{r^l Y_{l,m}^{0}(u,\phi)\} = \frac{1}{2}r^{l-1}\begin{bmatrix} u_x\{-Y_{l-1,m+1}^0 + (l+m)(l+m-1)Y_{l-1,m-1}^0\}\\ +u_y\{Y_{l-1,m+1}^e + (l+m)(l+m-1)Y_{l-1,m-1}^e\}\\ +u_z\{2(l+m)Y_{l-1,m}^0\}\end{bmatrix}. \tag{B7}$$

The derivatives on the right-hand side of (B1) are evaluated by using the following relations [25, p. 361]:

$$\frac{\partial}{\partial j}\left[r^l P_l^m(u)\begin{matrix}\cos\\ \sin\end{matrix} m\phi\right] = \begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\left[\frac{\partial}{\partial j}\{r^l P_l^m(u)e^{im\phi}\}\right] \tag{B2}$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)[r^l P_l^m(u)e^{im\phi}] = -r^{l-1}P_{l-1}^{m+1}(u)e^{i(m+1)\phi} \tag{B3}$$

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)[r^l P_l^m(u)e^{im\phi}] = (l+m)(l+m-1)r^{l-1}P_{l-1}^{m-1}(u)e^{i(m-1)\phi} \tag{B4}$$

$$\frac{\partial}{\partial z}[r^l P_l^m(u)e^{im\phi}] = (l+m)r^{l-1}P_{l-1}^m(u)e^{im\phi}. \tag{B5}$$

Following through with some tedious but straightforward algebra yields the following gradient formulae:

Appendix C
The u and ϕ Integrals Over the Surface of the Sphere

The surface harmonic functions, as defined in (10), can be shown to obey the following orthogonal properties. Unlike the standard spherical harmonics, these functions are not orthonormal. By direct substitution of the definitions for the functions $Y^e_{l,m}(u, \phi)$ and $Y^0_{l,m}(u, \phi)$, the following can be verified:

$$\int_{-1}^{+1}\int_0^{2\pi} Y^e_{l,m}(u, \phi)Y^e_{l',m'}(u, \phi)\, d\phi\, du = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\pi(1+\delta_{m,0})\delta_{m,m'}\delta_{l,l'} \quad \text{(C1)}$$

$$\int_{-1}^{+1}\int_0^{2\pi} Y^0_{l,m}(u, \phi)Y^0_{l',m'}(u, \phi)\, d\phi\, du = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\pi(1-\delta_{m,0})\delta_{m,m'}\delta_{l,l'} \quad \text{(C2)}$$

$$\int_{-1}^{+1}\int_0^{2\pi} Y^e_{l,m}(u, \phi)Y^0_{l',m'}(u, \phi)\, d\phi\, du = 0 \quad \text{(C3)}$$

where the orthogonality of associated Legendre polynomials [18], i.e.,

$$\int_{-1}^{+1} P_l^m(u)P_{l'}^m(u)du = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l,l'}$$

has been used.

Appendix D
The Skin Depth Functions $\Psi_l(z, t)$ and $g_l(q_n)$

Noting that

$$\cos\psi_{n,k,l} = -\frac{2q_n^2}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}},$$

$$\sin\psi_{n,k,l} = -\frac{x^2_{l+1/2,k}}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}}$$

the k-dependent terms in (2a) are

$$\sum_{k=1}^{\infty}\frac{J_{l+1/2}\left(x_{l+1/2,k}\dfrac{r}{R_s}\right)}{J_{l+1/2}(x_{l+1/2,k})}\frac{2q_n^2}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}}\cos(\omega_n t + \psi_{n,k,l})$$

$$= 2q_n^2\sum_{k=1}^{\infty}\frac{J_{l+1/2}\left(x_{l+1/2,k}\dfrac{r}{R_s}\right)}{J_{l+1/2}(x_{l+1/2,k})}\left[\frac{-2q_n^2\cos\omega_n t + x^2_{l+1/2,k}\sin\omega_n t}{4q_n^4 + x^4_{l+1/2,k}}\right]. \quad \text{(D1)}$$

The above sum is evaluated by expressing the modified Bessel function $I_{l+1/2}(zt)$ in a Fourier–Bessel series as

$$I_{l+1/2}(zs) = \sum_{k=1}^{\infty}\lambda_{l+1/2,k}(z)J_{l+1/2}(x_{l+1/2,k}s),$$

$$0 \le s \le 1, \qquad l = 0, 1, 2 \ldots, \quad \text{(D2)}$$

where z may be complex, $\lambda_{l+1/2,k}$ are yet-undetermined coefficients, and $x_{l+1/2,k}$ are defined in (4). Multiplying both sides of (D2) by $sJ_{l+1/2}(x_{l+1/2\cdot k'}s)$ and integrating from $s = 0$ to $s = 1$, gives

$$\lambda_{l+1/2,k} = \frac{2}{J^2_{l+1/2}(x_{l+1/2,k})}\cdot\int_{s=0}^{1} sJ_{l+1/2}(x_{l+1/2,k}s)I_{l+1/2}(zs)\, ds \quad \text{(D3)}$$

due to the orthogonality of the Bessel functions, i.e.,

$$\int_{s=0}^{1} sJ_{l+1/2}(x_{l+1/2,k}s)J_{l+1/2}(x_{l+1/2,k'}s)\, ds = \tfrac{1}{2}J^2_{l+1/2}(x_{l+1/2,k})\delta_{k,k'}.$$

The integral in (D3) can be evaluated to yield [21, (11.3.29)]

$$\int_{s=0}^{1} sJ_{l+1/2}(x_{l+1/2,k}s)I_{l+1/2}(zs)\, dt = e^{-i(\pi/2)(l-1)}\frac{J_{l+1/2}(x_{l+1/2,k})J_{l-1/2}(iz)}{z^2 + x^2_{l+1/2,k}}$$

and so

$$\lambda_{l+1/2,k} = \frac{2z}{J_{l+1/2}(x_{l+1/2,k})}\frac{1}{z^2 + x^2_{l+1/2,k}}I_{l-1/2}(z). \quad \text{(D4a)}$$

Therefore,

$$I_{l+1/2}(zs) = \sum_{k=1}^{\infty}\frac{2z}{z^2 + x^2_{l+1/2,k}}\frac{J_{l+1/2}(x_{l+1/2,k}s)}{J_{l+1/2}(x_{l+1/2,k})}I_{l-1/2}(z). \quad \text{(D4b)}$$

For $z = (1+i)q_n$, the above equation gives

$$\Psi_l(z, s) = \frac{1}{2z}\frac{I_{l+1/2}(zs)}{I_{l-1/2}(z)} = \sum_{k=1}^{\infty}\frac{J_{l+1/2}(x_{l+1/2,k}s)}{J_{l+1/2}(x_{l+1/2,k})}\left[\frac{-2iq_n^2 + x^2_{l+1/2,k}}{4q_n^4 + x^4_{l+1/2,k}}\right]. \quad \text{(D5)}$$

Finally, putting $s = r/R_s$, multiplying throughout by

$e^{j(\omega_n t + \pi/2)}$, and taking real parts leads to

$$\mathrm{Re}[\Psi(z, t)e^{j(\omega_n t + \pi/2)}]$$
$$= -\sum_{k=1}^{\infty} \frac{J_{l+1/2}\left(x_{l+1/2,k}\dfrac{r}{R_s}\right)}{J_{l+1/2}(x_{l+1/2,k})} \cdot \left[\frac{-2q_n^2 \cos\omega_n t + x_{l+1/2,k}^2 \sin\omega_n t}{4q_n^4 + x_{l+1/2,k}^4}\right]. \quad \text{(D6)}$$

Together, (D1), (D6), and (2a) establish (5a) and (5b). Finally, for $z = (1 + i)q_n = s = 1$, equating real parts throughout (D5) gives

$$\sum_{k=1}^{\infty} \frac{x_{l+1/2,k}^2}{4q_n^4 + x_{l+1/2,k}^4} = \mathrm{Re}\left[\frac{1}{2z}\frac{I_{l+1/2}(z)}{I_{l-1/2}(z)}\right] = \mathrm{Re}[\Psi_l(z, 1)]. \quad \text{(D7)}$$

To prove (19b), consider the function $s^{l+1/2}$. Since it is continuous in the interval $0 \le s \le 1$, it can be expanded in a Fourier–Bessel series as

$$s^{l+1/2} = \sum_{k=1}^{\infty} \lambda_{l+1/2,k} J_{l+1/2}(x_{l+1/2,k}s). \quad \text{(D8)}$$

As before, the undetermined coefficients $\lambda_{l+1/2,k}$ are determined by making use of the orthogonality of Bessel functions. Thus,

$$\lambda_{l+1/2,k} = \frac{2(2l+1)}{x_{l+1/2,k}^2 J_{l+1/2}(x_{l+1/2,k})} \quad \text{(D9a)}$$

and

$$s^{l+1/2} = 2(2l+1)\sum_{k=1}^{\infty} \frac{J_{l+1/2}(x_{l+1/2,k}s)}{x_{l+1/2,k}^2 J_{l+1/2}(x_{l+1/2,k})}. \quad \text{(D9b)}$$

For $s = 1$,

$$\sum_{k=1}^{\infty} \frac{1}{x_{l+1/2,k}^2} = \frac{1}{2(2l+1)}. \quad \text{(D10)}$$

Finally, the subtraction of (D10) from (D7) gives the desired result, i.e., (19b).

The limiting case in (21a) can be proved by rewriting (19b) as

$$g_l(q_n) = \sum_{k=1}^{\infty} \frac{1}{x_{l+1/2,k}^2} \frac{-1}{1 + \left(\dfrac{x_{l+1/2,k}^4}{4q_n^4}\right)}. \quad \text{(D11)}$$

By virtue of (D10), (21a) follows. Similarly, in the limit of very small q_n, (19b) can be written as

$$\underset{q_n \to \infty}{Lt}[g_l(q_n)] = 4q_n^4 \sum_{k=1}^{\infty} \frac{-1}{x_{l+1/2,k}^6}. \quad \text{(D12)}$$

Multiply (D9b) by $s^{l+3/2}$ and integrate from $s = 0$ to $s = s$. The use of (A6a) leads to

$$s^{l+3/2} = 2(2l+1)(2l+3)\sum_{k=1}^{\infty} \frac{J_{l+3/2}(x_{l+1/2,k}s)}{x_{l+1/2,k}^3 J_{l+1/2}(x_{l+1/2,k})}.$$

Multiply the above equation, once again, by $s^{l+5/2}$ and integrate from $s = 0$ to $s = s$. This leads to

$$s^{l+5/2} = 2(2l+1)(2l+3)(2l+5) \cdot \sum_{k=1}^{\infty} \frac{1}{x_{l+1/2,k}^4} \frac{J_{l+5/2}(x_{l+1/2,k}s))}{J_{l+1/2}(x_{l+1/2,k})}. \quad \text{(D13)}$$

Putting $s = 1$ in (D13), and then using the recursion relation (A6b) in conjunction with (4), gives

$$1 = 2(2l+1)(2l+3)(2l+5) \cdot \sum_{k=1}^{\infty}\left[\frac{(2l+1)(2l+3)}{x_{l+1/2,k}^6} - \frac{1}{x_{l+1/2,k}^4}\right]. \quad \text{(D14)}$$

Squaring both sides of (D8) and then multiplying throughout by s, and finally integrating from $s = 0$ to $s = 1$ (Parseval's theorem), yields

$$\int_0^1 z^{2l+2}\,dz = \sum_{k=1}^{\infty}\sum_{k'=1}^{\infty} \lambda_{l+1/2,k}\lambda_{l+1/2,k'} \cdot \int_0^1 z J_{l+1/2}(x_{l+1/2,k}z) J_{l+1/2}(x_{l+1/2,k'}z)\,dz.$$

By using the orthogonality of Bessel functions and then (D9a) results in

$$\sum_{k=1}^{\infty} \frac{1}{x_{l+1/2,k}^4} = \frac{1}{2(2l+3)(2l+1)^2}. \quad \text{(D15)}$$

Together, (D15) and (D14) give (21b).

Appendix E
The Difference Between the Homogeneous and Nonhomogeneous Models

The magnetic field on the axis of a circular loop is purely z-directed and is given by [19]

$$\boldsymbol{B}_o(z_o) = \boldsymbol{u}_z \mu_o I_o \frac{1}{2}\frac{a^2}{(a^2 + z_o^2)} \quad \text{(E1)}$$

where z_o is the axial distance from the center of the loop of radius a. Using superposition and (28), the homogeneous model yields the following expression for the Lorentz force[2] on the sphere along the axis of the stack of loops shown in Fig. 3:

$$\left.\frac{\boldsymbol{F}_s}{\mu_o I_o^2}\right|_{\mathrm{hom}} = \pi \overline{R}_s^3 \sum_n \sum_{n'} G(q) \frac{t_n^2 t_{n'}^2 s_{n'}}{(t_n^2 + s_n^2)^{3/2}(t_{n'}^2 + s_{n'}^2)^{5/2}} \quad \text{(E2)}$$

where $t_n = 1 + \gamma c_n$ and $s_n = \overline{z} - \gamma c_n$. For the special case of a single loop, the force on the sphere at a height z_o above the center of the loop becomes

$$\left.\frac{\boldsymbol{F}_s}{\mu_o I_o^2}\right|_{\mathrm{hom}} = \pi R_s^3 G(q) \frac{a_1^4 z_o}{(a_1^2 + z_o^2)^4}. \quad \text{(E3)}$$

[2]Assuming that all loops carry current at the same frequency.

If the sphere is small compared to the radius of the loop, then all terms of order $l \geq 2$ in (26a) may be dropped. Considering only the $l = 1$ term and expressing all functions in terms of the dimensions in Fig. 3, (26a) can be written as

$$\left.\frac{F_s}{\mu_o I_o^2}\right|_{\substack{l=1\\ \text{non-hom}}} = \frac{9}{2}\,\pi R_s^3 g_1(q)\,\frac{a_1^4 z_o}{(a_1^2 + z_o^2)^4}. \tag{E4}$$

By (20b), it follows that

$$\left[\left.\frac{F_s}{\mu_o I_o^2}\right|_{\substack{l=1\\ \text{non-hom}}} - \left.\frac{F_s}{\mu_o I_o^2}\right|_{\text{hom}}\right] \Bigg/ \left[\left.\frac{F_s}{\mu_o I_o^2}\right|_{\substack{l=1\\ \text{non-hom}}}\right] = -\frac{1}{3}. \tag{E5}$$

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