Nonlinear Coupled Thermo-Electromagnetic Problems With the Cell Method

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An application of the Cell Method to a coupled nonlinear problem is presented. The two coupled physical aspects consist of the magnetic quasistatic and thermal transient problems. Their investigation is concerned in a wide range of applications such as the analysis of devices for the induction heating treatment of conductive workpieces. Assuming a cylindrical geometry, that is usual in such devices, the numerical model exploits a two-dimensional axis-symmetrical formulation to perform the simultaneous analyses of the transient thermal and ac eddy currents problems. As a consequence of the large temperature rise, typical in these treatments, both coupled problems are considered nonlinear. The formulation is validated comparing the numerical results against those adopted with commercial codes based on the finite element method and against the experimental data measured on a test device, too.

Index Terms—**Cell method, coupled problems, eddy currents, finite formulations, induction-heating.**

I. Introduction

THE CELL METHOD has originally been proposed by Enzo Tonti [1] and as long as some years it is gaining growing interest for the computation of electromagnetic fields. It presents a lot of affinities with the Finite Integral Technique [2] and similarly to this one, it does not require any notions of differential nature such as gradient, curl, divergence, but considers the electromagnetic field equations in finite (namely "integral," in the differential Maxwell's approach) quantities, directly written in algebraic finite formulation. The theoretical basis of the method is the classification of field quantities into *configuration quantities* (describing the field configuration), *source quantities* (originating the field) and *energy quantities* (arising basically from the interaction of a configuration quantity and a source quantity). This classification of finite quantities call for a geometrical subdivision of the field space based on algebraic topology notions that produces two sets of cell complexes (oriented points, lines, surfaces and volumes), staggered one to each other.

Configuration quantities are defined on the "primal" cell complex K, whose domains are all endowed with inner orientation (volumes $\mathbf{V}$, surfaces $\mathbf{S}$, lines $\mathbf{L}$ and points $\mathbf{P}$), while source quantities are related to the "dual" cell complex, whose domains are all endowed with outer orientations ($\tilde{\mathbf{V}}\tilde{\mathbf{S}}\tilde{\mathbf{L}}\tilde{\mathbf{P}}$).

In the finite approach, configuration quantities are related to each other by nonmetric structure equations and source quantities are related to each other by nonmetric structure equations as well, whilst configuration and source quantities are linked together by metric constitutive equations containing the medium physical parameters.

During the last two years both the time harmonic eddy currents and the transient thermal nonlinear problems in axis-symmetrical geometries have been independently investigated by the research group at Padua, resorting to the Cell Method [3]–[5]. In this paper, we present an application where the two aspects are investigated together, in the contest of coupled eddy currents and the thermal nonlinear problems. One initial assumption is to take into account possible strong coupling of the two problems, that requires electromagnetic and thermal quantities to be computed simultaneously. These requirements are not needed in many industrial induction heating applications, thanks to very different electromagnetic and thermal time constants. But our target is to develop an algorithm able to face more general conditions. In particular a further version is under development devoted to study electro-thermal problems in semiconductor structures, where the two typical times are actually comparable [6].

II. Algebraic Formulation

A. Differential Formulation

In the differential formulation the coupled time harmonic eddy currents and transient thermal problems in axis-symmetrical conditions can be described by the following equations:

$$\begin{cases} \nabla \times (\nu \nabla \times \dot{A}) + j\omega\sigma\dot{A} = \dot{J}_s \\ \nabla \cdot (-\lambda \nabla T) + \gamma c \partial_t T = w \end{cases} \tag{1}$$

where $\dot{J}$ is the peak values phasor of the impressed current density with azimuthal direction, $\dot{A}$ is the phasor of the corresponding magnetic vector potential, T the temperature, ω the angular frequency, $\upsilon, \sigma, \lambda, \gamma$, and c the media parameters (magnetic reluctivity, electric conductivity, thermal conductivity, specific heat, and density, respectively). w is the Joule losses power density, namely the source of the thermal diffusion, that couples the two problems and can be expressed in terms of the vector potential $\dot{A}$ as

$$w = \frac{1}{2}\sigma\omega^2\dot{A}\dot{A}^* \tag{2}$$

where $*$ stands for conjugate. The coupling between the two physical aspects also depends on the material coefficients $\upsilon, \sigma, \lambda, \gamma$ and c, whose T dependence for large temperature rises must be taken into account, thus causing the problem to be nonlinear.

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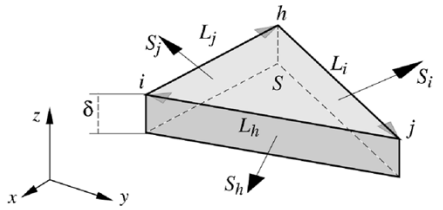

Fig. 1. Space tessellation in the case of planar symmetry with the cartesian coordinate system: primal prismatic volume cell (3-cell) with its surfaces (2-cells), lines (1-cells) and points 0-cells), used for both the electromagnetic and thermal problems; its trace in the 2D x, y plane is the simplicial triangle with vertices h, i, j. In the case of cylindrical symmetry the depth δ is proportional to the radius r.

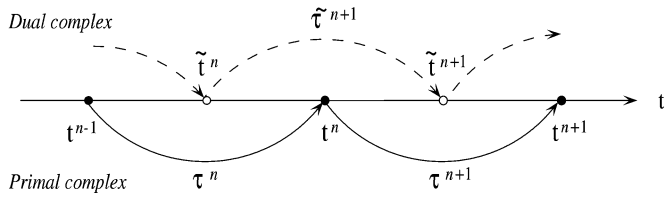

Fig. 2. Time tessellation with barycentric primal and dual intervals (1-cells) and instants (0-cells), needed for the transient thermal problem evolution.

B. CM Finite Formulation

The space tessellation used in order to re-formulate both the electromagnetic and thermal problems of (1) in finite form according to the Cell Method is based on the prismatic primal volumes shown in Fig. 1, whose trace in the 2D domain are triangles with vertices h, i, j. The transient thermal problem requires also a time tessellation in primal and dual complexes of time intervals and instants (1-cells and 0-cells): a barycentric choise has been adopted, as shown in Fig. 2. The CM formulation of (1) on such complexes leads to the following system of two matrix equations, already described in detail in previous papers [3], [4]:

$$\begin{cases} (\boldsymbol{\dot{D}} + j\omega\boldsymbol{\dot{C}})\boldsymbol{\dot{P}} = \boldsymbol{\dot{I}}_S \\ \left(\frac{1}{\tau}\boldsymbol{G} + \frac{1}{2}\boldsymbol{F}\right)\boldsymbol{T}^n = \left(\frac{1}{\tau}\boldsymbol{G} - \frac{1}{2}\boldsymbol{F}\right)\boldsymbol{T}^{n-1} + \frac{1}{2}(\boldsymbol{W}^n + \boldsymbol{W}^{n-1}) \end{cases} \tag{3}$$

with $\boldsymbol{\dot{P}}$ the electro-kinetic momentum vector (namely the line integral of the vector potential along primal 1-cells with azimuthal direction), $\boldsymbol{T}$ the nodal temperature vector (in primal 0-cells), $\boldsymbol{D}$ and $\boldsymbol{C}$ the stiffness and mass matrices of the electromagnetic problem, $\boldsymbol{G}$ and $\boldsymbol{F}$ the mass and stiffness matrices of the thermal problem, $\boldsymbol{\dot{I}}_S$ the vectors of the impressed currents. The last term in the second equation of (3) is the vector of the Joule losses power in each 3-cell at the dual instants $\tilde{t}^n$: it is computed as average of the values at the nearest primal instants, namely $\boldsymbol{W}^{n-1}$ and $\boldsymbol{W}^n$, deriving from the time barycetric tessellation of Fig. 2. In this way the time integration algorithm for the temperature evolution corresponds to an application of the Crank–Nicolson time scheme.

The matrices $\boldsymbol{D}, \boldsymbol{C}, \boldsymbol{G}$, and $\boldsymbol{F}$ are assembled from the corresponding local matrices which in turn are computed making use of the affine local coordinates shown in Fig. 3, [7], which allow to obtain the same matrix coefficients for all primal cells. For instance, the local matrix for $\boldsymbol{C}$ is

$$\boldsymbol{C}_\ell = \frac{\sigma\tilde{S}}{36r}\begin{bmatrix} 22 & 7 & 7 \\ 7 & 22 & 7 \\ 7 & 7 & 22 \end{bmatrix} \tag{5}$$

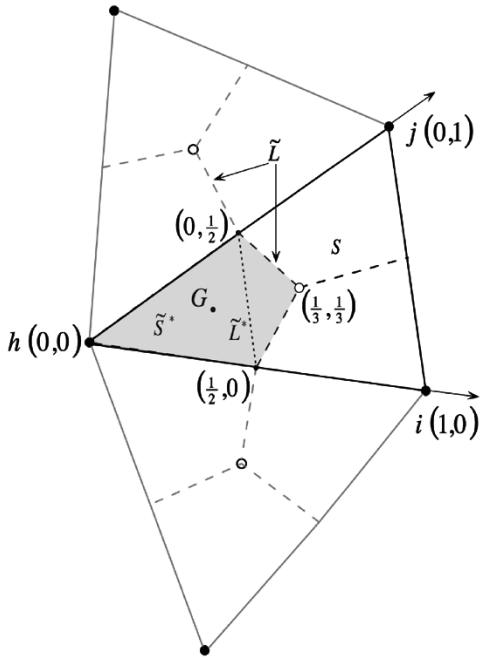

Fig. 3. Affine local coordinates used for the construction of the normalized local matrices for $\boldsymbol{D}, \boldsymbol{C}, \boldsymbol{G}$, and $\boldsymbol{F}$.

full deduction of which have been given in previous papers [3] and [4]; $\tilde{S}$ is the dual 3-cell surface and r its mean radius, namely the normalized azimuthal cell depth.

The Joule losses power, being the source quantity for the thermal problem, is computed on dual 3-cell, namely volumes whose trace in the 2-D domain are dual surfaces surrounding each primal node ($\tilde{S}_h$ for the point of index h). In the present finite formulation, each of them is obtained from the electro-kinetic momentum $\dot{P}_h$ along the primal line through the point of index h and normal to the 2-D domain

$$W = \frac{1}{2}\frac{\sigma\tilde{S}_h}{r_h}\omega^2\dot{P}_h\dot{P}_h^* \tag{6}$$

corresponding to the differential local expression (2), where $G_h = \sigma\tilde{S}_h/r_h$ is the normalized conductance of the element having normalized azimuthal cell depth r_h, along which the fasorial induced voltage is $\omega\dot{P}_h$.

In order to take into account possible strong coupling between the electromagnetic and thermal problems, the simultaneous solution of both equations in (3) has been imposed, leading to the following recursive nonlinear system:

$$\begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & \boldsymbol{N} \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{P}} \\ \boldsymbol{T}^n \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{K} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{I}}_S \\ \boldsymbol{U} \end{bmatrix} \tag{7}$$

with

$$\begin{cases} \boldsymbol{M} = \boldsymbol{D} + j\omega\boldsymbol{C} \\ \boldsymbol{N} = \frac{1}{\tau}\boldsymbol{G} + \frac{1}{2}\boldsymbol{F} \\ \boldsymbol{K} = -\frac{1}{2}\boldsymbol{W}^n \\ \boldsymbol{U} = \left(\frac{1}{\tau}\boldsymbol{G} - \frac{1}{2}\boldsymbol{F}\right)\boldsymbol{T}^{n-1} + \frac{1}{2}\boldsymbol{W}^{n-1} \end{cases}. \tag{8}$$

C. Time Integration Scheme

In order to solve (7) at every time step, a Newton–Raphson iterative algorithm has been adopted. At this aim, the Jacobian of the fundamental matrix must be computed, that is included in the following linear equations system with unknown vector

$[\Delta\dot{\boldsymbol{P}}\ \Delta\boldsymbol{T}]^{\mathrm{T}}$, containing the iterative increments of the value at $[\dot{\boldsymbol{P}}\ \boldsymbol{T}^n]^{\mathrm{T}}$ every time step:

$$\begin{bmatrix} \frac{\partial(\boldsymbol{M}\dot{\boldsymbol{P}})}{\partial\dot{\boldsymbol{P}}} & \frac{\partial(\boldsymbol{M}\dot{\boldsymbol{P}})}{\partial\boldsymbol{T}} \\ \frac{\partial(\boldsymbol{N}\boldsymbol{T}+\boldsymbol{K})}{\partial\dot{\boldsymbol{P}}} & \frac{\partial(\boldsymbol{N}\boldsymbol{T}+\boldsymbol{K})}{\partial\boldsymbol{T}} \end{bmatrix} \begin{bmatrix} \Delta\dot{\boldsymbol{P}} \\ \Delta\boldsymbol{T} \end{bmatrix} = -\boldsymbol{R} \tag{9}$$

with $\boldsymbol{R}$ the residual vector of (7) in the iterative approach. Assuming time step constant values for the physical parameters $\mu, \sigma, \lambda, \gamma$, and c over each time interval, (9) reduces to

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ -\frac{1}{2}\frac{\partial\boldsymbol{W}}{\partial\dot{\boldsymbol{P}}} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \Delta\dot{\boldsymbol{P}} \\ \Delta\boldsymbol{T} \end{bmatrix} = -\boldsymbol{R}. \tag{10}$$

The Jacobian matrix of (10) is assembled by means of a cycle on all primal cell, that makes use of the local submatrices. The construction procedure of these local submatrices $\boldsymbol{M}_\ell$ and $\boldsymbol{N}_\ell$ resorts again of the affine local coordinates of Fig. 3, which provide the same matrix coefficients for all primal cells [3], [4].

The local submatrix $\partial\boldsymbol{W}_\ell/\partial\dot{\boldsymbol{P}}$ is diagonal, being related to a primal cell whose trace in the 2-D domain is the triangle with vertices h, i, j of Fig. 1. It has been approximated by the matrix:

$$\frac{\partial\boldsymbol{W}_\ell}{\partial\dot{\boldsymbol{P}}} = \begin{bmatrix} \frac{\partial W_h}{\partial P_h} & 0 & 0 \\ 0 & \frac{\partial W_i}{\partial P_i} & 0 \\ 0 & 0 & \frac{\partial W_j}{\partial P_j} \end{bmatrix} \tag{11}$$

with

$$\begin{cases} \frac{\partial W_h}{\partial P_h} = \frac{\sigma\tilde{S}_h}{r_h}\omega^2 P_h \\ \frac{\partial W_i}{\partial P_i} = \frac{\sigma\tilde{S}_i}{r_i}\omega^2 P_i \\ \frac{\partial W_j}{\partial P_j} = \frac{\sigma\tilde{S}_j}{r_j}\omega^2 P_j \end{cases}. \tag{12}$$

III. Numerical and Experimental Validation

This algorithm for the solution of coupled electromagnetic and thermal problems has been developed as a general purpose tool, possibly with comparable electromagnetic and thermal time constants, as can happen in a wide class of semiconductive media, so involving a strong coupling among the two physical aspects [6]. Nevertheless during first validation stage it has been applied to the numerical simulation of industrial devices for induction heating treatments, for which experimental data were soon available at the Laboratory of Electro-Heat (LEP) of the Electrical Engineering Department of Padua University.

In order to perform the experimental test validating the CM algorithm, the simple induction heating apparatus shown in Fig. 4 has been set up. It consists of a 32 turn coil with a magnetic steel billets inserted into its hole. The geometrical dimensions are shown in Fig. 5. In order to provide the adiabatic boundary condition to the cylinder, a thermal insulation cover not show in Fig. 4 has been used. The coil has been fed at 300 A at 2.8 kHz and several thermocouples have been applied on the cylinder surface in order to measure the billet temperature at different axial positions on the cylinder surface.

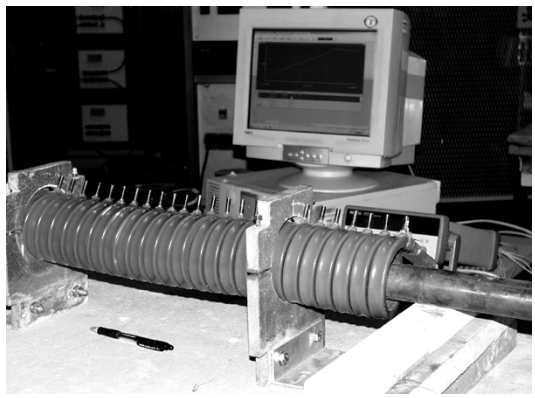

Fig. 4. Experimental setup used for validation. In the picture, the workpiece is shown inside the inductor without its thermal insultaing cover, used during measure tests. The workpiece is made of steel and the inductor consists of 32 turns fed at 300 A at 2.8 kHz.

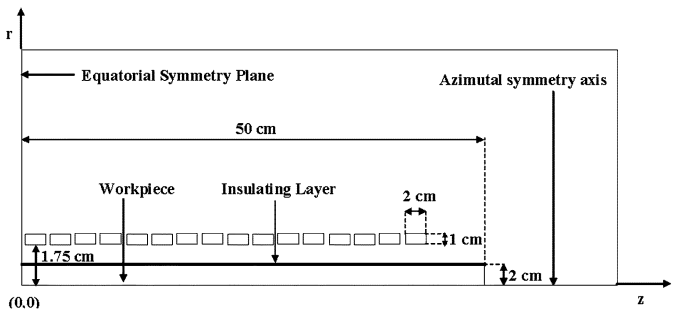

Fig. 5. Geometrical dimensions of the experimental setup used for validation.

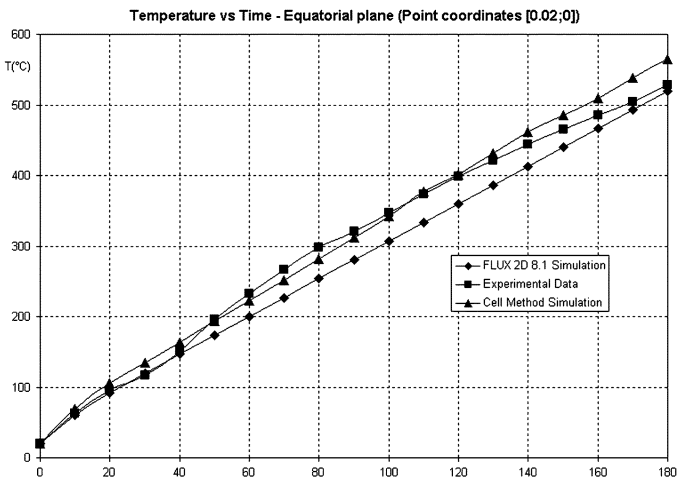

Fig. 6. Comparison among the experimental temperature measured on the apparatus of Fig. 4 and the corresponding numerical results obtained with the CM algorithm and the FEM code. The test point is on the billet surface at the equatorial plane.

The billet has been heated up to 500 °C measuring the main electrical quantities and the temperature. A comparison has also carried out with the results obtained from a commercial FE code (Flux 2D), able to cope with the electromagnetic and thermal coupled problems, taking into account the temperature dependency of the electric and thermal material parameters.

Fig. 6 shows a comparison among temperature data from experimental device, FE and CM codes in a point on the equatorial plane. Fig. 7 shows a similar comparison between temperature data from FE and CM codes in the hottest point. As can

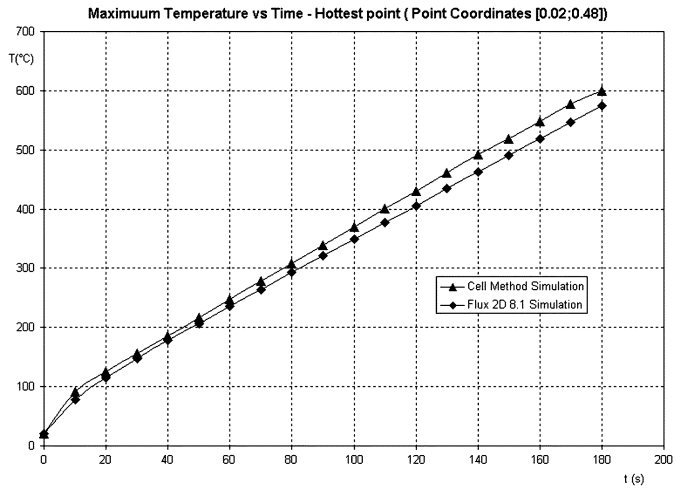

Fig. 7. Comparison between the numerical temperature results obtained with the CM algorithm and the FEM code in the hottest point on the billet.

be seen, discrepancies within 10% are encountered. At this concern, some comments must be pointed out. First, the temperature dependence of the physical parameters of the C40 steel used in the test is not known with high accuracy: some reasonable assumptions had to be adopted. Further, both numerical codes assume a constant rms coil current, whilst in the experimental apparatus it presented some variation, due to voltage control of the power supply. Another reason of difference is related to the coil current distribution between the FE and Cell codes. In the former an external circuit is used and the total current is imposed accounting for a nonuniform current density distribution in the coil turns; in the second a uniform current density has been imposed in each turn. Taking into account all these statements, we can say that the results obtained are satisfactory. It must also be noted that while Flux adopts a second order approximation, the present CM algorithm makes use of a first order one. To this regard, a second order CM procedure is under development in our Lab and a related paper is referred in [8].

IV. Conclusion

The Cell Method proves to be a interesting and promising formulation for dealing with coupled numerical problems encountered in technical and industrial applications, due to its simple theoretical formulation. The implementation here presented accounts for strong thermal and electromagnetic coupling, and the results obtained up to now suggest that future development can provide wider simulation capabilities. But more work remains in order to achieve a full operational package, able to deal with a wide range of problems encountered in industrial applications such as induction melting, structural stress investigations in electrical machines, or thermal stress in semiconductor structures.

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