An Efficient Time Domain Method for Nonlinear Periodic Eddy Current Problems

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A time-domain method is presented yielding the periodic steady-state solution of nonlinear eddy current problems without having to step through the transient process. A novel technique is introduced which, under periodic conditions, in the linear case, decouples the finite element equation systems written for different time steps. Thereupon, a fixed point method is used to iteratively solve the nonlinear equations. The resulting procedure retains the decoupling property valid in the linear case, therefore, it suffices to step through one period only. The efficiency of the method is illustrated by two two-dimensional examples.

***Index Terms*—Eddy currents, nonlinear equations, periodic functions, time-domain analysis.**

I. Introduction

ONE of the most challenging problems in computational electromagnetics is the efficient computation of steady-state solutions of nonlinear time-periodic problems. The most straightforward treatment is the so called "brute force" method of using time stepping technique starting from arbitrary (usually zero) initial values (see, e.g., [1]). This may require stepping through several periods and is, therefore, computationally expensive. An improvement can be achieved by using approximate frequency domain techniques to estimate the initial value [2]. An important development is the time-periodic finite element method which applies a special version of the Newton–Raphson procedure [3]. Only time steps within one period are introduced, and the number of unknowns in each linear system is just the number of degrees of freedom at a time instant, but additional iteration steps are necessary to enforce periodicity. It has also been introduced in the analysis of three-dimensional (3-D) eddy current problems [4]. A further frequently used method is based on the idea of harmonic balance. An early finite element realization is found in [5], and a 3-D eddy current application in [6]. The harmonics are coupled, therefore, additional iterations have to be used to avoid having to solve for all harmonics simultaneously.

In this paper, a novel technique applicable in linear case is introduced first which decouples the equations relating to different time steps within one period with the periodicity enforced. Thereupon, a fixed point method inherently resembling the linear case is used to solve the nonlinear system. This results in computational costs corresponding to time stepping through one period only, a potentially optimal solution.

II. Periodic Linear Case

Applying finite element Galerkin techniques to any time domain formulation of linear eddy current problems results in a set of ordinary differential equations

$$Sx(t) + M\dot{x}(t) = f(t) \tag{1}$$

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where t is time, $x(t)$ is the vector of unknowns, $\dot{x}(t)$ is its time derivative, $f(t)$ is a known vector, and S and M are large, sparse square matrices. The order of these vectors and matrices equals the number of degrees of freedom. If the forcing vector is time periodic with a period T, i.e., $f(t) = f(t+T)$, then there exists a periodic steady-state solution satisfying the periodicity condition $x(0) = x(T)$.

The time discretization of (1) within one period with a constant time step $\Delta t = T/N$ leads to N coupled simultaneous linear equation systems

$$Ax_k + Bx_{k-1} = b_k, \quad k = 1, 2, \ldots, N \tag{2}$$

where A and B are linear combinations of the matrices S and M, b_k is a linear combination of the known vectors $f(k\Delta t)$ and $f((k-1)\Delta t)$, and $x_k = x(k\Delta t)$ is the vector of unknowns at the kth time step. The steady-state periodic solution can be obtained by enforcing the condition

$$x_0 = x_N. \tag{3}$$

Multiplying the kth equation in (2) by $e^{j(i-1)(k-1)(2\pi/N)}, i = 1, 2, \ldots, N$ and adding all equations results in N uncoupled equation systems

$$\left(A + e^{j(i-1)\frac{2\pi}{N}}B\right)y_i = c_i, \quad i = 1, 2, \ldots, N \tag{4}$$

where

$$y_i = \sum_{n=1}^{N} e^{j(n-1)(i-1)\frac{2\pi}{N}} x_n, \quad i = 1, 2, \ldots, N \tag{5}$$

$$c_i = \sum_{n=1}^{N} e^{j(n-1)(i-1)\frac{2\pi}{N}} b_n, \quad i = 1, 2, \ldots, N. \tag{6}$$

The computational effort needed for the solution of the N equation systems in (4) is comparable to solving (2) for N time steps (i.e., for one period only), with the slight complication of having to use complex arithmetics. Once this has been carried out, the time values x_k can be directly expressed from the solutions y_i. Indeed, multiplying the ith equation in (5) by

$e^{-j(i-1)(k-1)(2\pi)/(N)}, k = 1,2,\ldots,N$ and adding all equations results in

$$\sum_{n=1}^{N}\left(\sum_{i=1}^{N} e^{-j(i-1)(k-n)\frac{2\pi}{N}}\right)x_n = \sum_{i=1}^{N} e^{-j(i-1)(k-1)\frac{2\pi}{N}} y_i, \quad k = 1,2,\ldots,N. \tag{7}$$

Now, using the identities

$$\sum_{i=1}^{N} e^{-j(i-1)(k-n)\frac{2\pi}{N}} = \begin{cases} 0, & \text{if } n = 1,2,\ldots,k-1,k+1,\ldots,N \\ N, & \text{if } n = k \end{cases} \tag{8}$$

results in

$$x_k = \frac{1}{N}\sum_{i=1}^{N} e^{-j(i-1)(k-1)\frac{2\pi}{N}} y_i, \quad k = 1,2,\ldots,N. \tag{9}$$

The desired time values of the solution $x(t)$ at the time instants $t_k = k\Delta t$ are hence simple linear combinations of the solutions y_i of the N uncoupled equation systems in (4). In summary, the steady-state periodic solution of *linear* eddy current problems with time periodic excitation can be solved at the cost of time stepping through one period only.

III. Periodic Nonlinear Case

A. Flux Density Based Formulations

Let us first assume that the formulation leading to (1) is based on the flux density, $\mathbf{B}$. This is the case if $\mathbf{B}$ is described by the magnetic vector potential, $\mathbf{A}$. If the relationship between the magnetic field intensity, $\mathbf{H}$ and $\mathbf{B}$

$$\mathbf{H} = \mathbf{H}(\mathbf{B}) \tag{10}$$

is nonlinear, the matrix S in (1) and, hence, the matrices A and B in (2) depend on x_k, so the Fourier block-diagonalization described in the previous section does not work. In such a case, it is advantageous to use a fixed point method to solve the nonlinear (2) (see, e.g., [7]). This means writing (10) as

$$\mathbf{H} = \nu_{\mathrm{FP}}\mathbf{B} - \mathbf{M}_{\mathrm{FP}} \tag{11}$$

where ν_{FP} is an appropriate reluctivity independent of $\mathbf{B}$ and $\mathbf{M}_{\mathrm{FP}}$ is a field dependent magnetization-like quantity. Starting from an arbitrary $\mathbf{M}_{\mathrm{FP}}$, the fixed point method means updating it at each iteration step. The value of the fixed point reluctivity ν_{FP} can be chosen so that the method is convergent [7]. The rate of convergence depends on the choice of the reluctivity, an optimal value is given in [7]. Overrelaxation can substantially accelerate convergence [7], [8].

Using the same value for ν_{FP} at every time step, the matrices A and B in (2) are the same for all values of k, so that the block-diagonalization described in Section II can be applied at each nonlinear iteration step. As a result, the computational effort is comparable to time stepping through one period only.

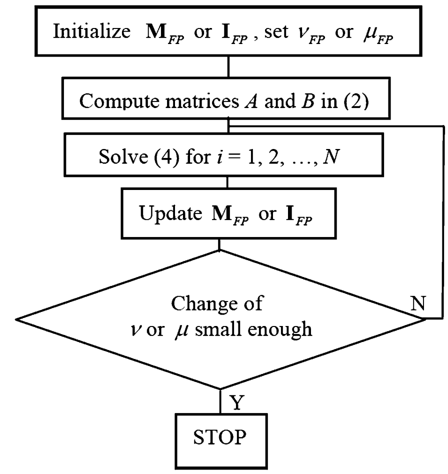

Fig. 1. Flowchart of the proposed method.

B. Field Intensity Based Formulations

If the formulation leading to (1) is based on the magnetic field intensity, $\mathbf{H}$, e.g., when using the electric vector potential, $\mathbf{T}$ and the magnetic scalar potential, Φ to describe the magnetic field, and the relationship between $\mathbf{B}$ and $\mathbf{H}$

$$\mathbf{B} = \mathbf{B}(\mathbf{H}) \tag{12}$$

is nonlinear, then (1) has to be modified to

$$Sx(t) + \frac{d}{dt}[Mx(t)] = f \tag{13}$$

with the matrix M depending on $x(t)$. Hence, the matrices A and B in (2) will depend on both x_k and x_{k-1} and the block-diagonalization technique does not work, unless this dependence is eliminated by a suitable choice of the nonlinear technique employed. The fixed point method helps again. It involves replacing (12) by

$$\mathbf{B} = \mu_{\mathrm{FP}}\mathbf{H} + \mathbf{I}_{\mathrm{FP}} \tag{14}$$

where μ_{FP} is a suitable permeability independent of $\mathbf{H}$ and $\mathbf{I}_{\mathrm{FP}}$ is a field dependent polarization-like quantity. $\mathbf{I}_{\mathrm{FP}}$ is updated at each nonlinear iteration step. The value of μ_{FP} can be chosen to ensure the convergence of the method [7]. Techniques to accelerate the convergence are discussed in [7] and [8].

If μ_{FP} is chosen to have the same value at each time step, then the matrices A and B do not depend on x_k or x_{k-1} and, therefore, the Fourier block-diagonalization method of Section II is valid for each nonlinear iteration step. Again, this means that the computational effort necessary is about the same as that required if time stepping is employed through just one period.

The flowchart of the proposed method is shown in Fig. 1.

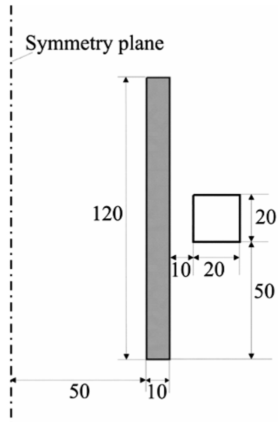

Fig. 2. Aluminum conductor ($\sigma = 3.510^7$ S/m) with a sinusoidal voltage of 0.66 V/m peak, 50 Hz parallel to a steel wall ($\sigma = 1.0 \times 10^7$ S/m).

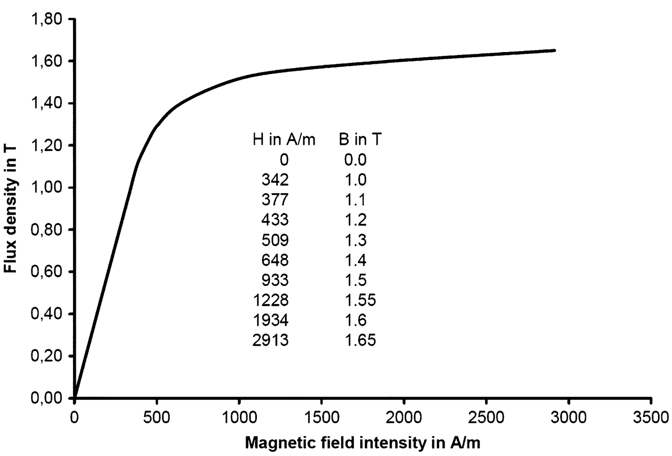

Fig. 3. B–H curve of ferromagnetic wall and of ferromagnetic screen.

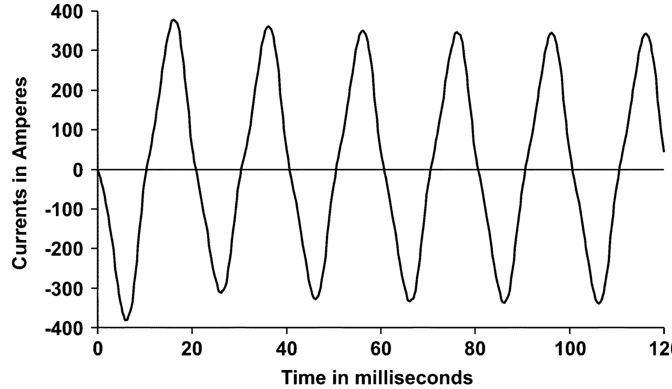

Fig. 4. Total current through ferromagnetic wall by time stepping.

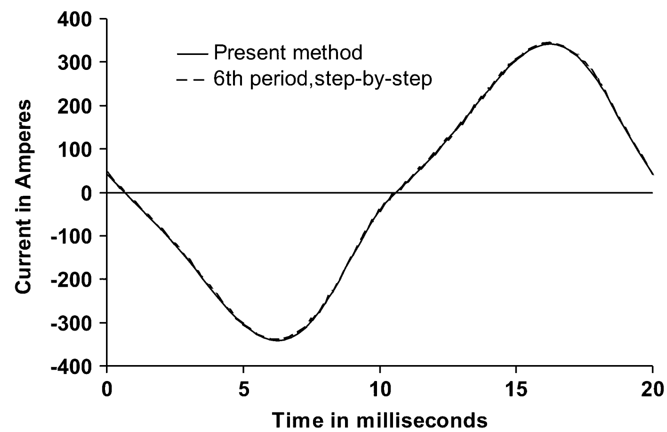

Fig. 5. Comparison of total current through ferromagnetic wall by present method and by step-by-step method.

IV. Numerical Examples

A. Conductor Near Conducting Ferromagnetic Wall

This two-dimensional (2-D) example involves an aluminum conductor with given sinusoidal voltage parallel to a saturated ferromagnetic wall (Fig. 2). The nonlinear B–H curve of the wall is shown in Fig. 3. The problem has been solved by the “brute force” method with six periods stepped through, practically arriving at the steady-state solution. One period has been discretized in 20 time steps. The nonlinear equations have been solved using a robust direct iteration technique [1] until the mean relative variation of the reluctivity over the Gaussian integration points fell below 0.1% and the maximal relative variation below 1%. The number of linear equation systems to be solved is 1139. The time variation of the total current through the ferromagnetic wall is shown in Fig. 4.

Using the present method, the same stopping criterion for the nonlinear iterations has been applied as in the step-by-step approach. No overrelaxation has been employed, the value of the relative permeability corresponding to ν_{FP} has been taken to be 490. No more than 210 linear equation systems had to be solved to step through the 20 time steps. The resulting current is shown in Fig. 5 to coincide with the current in the sixth period computed by the “brute force” method.

B. Conductor Shielded by Conducting Ferromagnetic Screen

A second 2-D example comprises a copper conductor with a given sinusoidal voltage within a conducting ferromagnetic screen (Fig. 6). The B–H curve of the screen is again the one shown in Fig. 3. Due to the much larger time constant of the transients, the “brute force” method has required 60 periods to be stepped through in order to practically arrive at steady-state. Each period has again been discretized in 20 time steps. The stopping criterion of the nonlinear direct iterations according to [1] has been 0.1% for the mean relative variation of the reluctivity in the integration points and 1% for the maximal relative variation. All in all, 19 058 linear equations have had to be solved. The time variation of the current in the copper conductor is shown in Fig. 7.

The stopping criterion for the nonlinear fixed point iterations of the paper has been the same as in the brute force method. Again, no overrelaxation has been used and the relative permeability corresponding to ν_{FP} has been chosen to be 450. The number of linear equations to be solved turned out to be 980 for the 20 time-steps. The resulting current in the conductor is compared with the time variation in the 60th period obtained by time-stepping in Fig. 8. Notice that the current by the brute force method after 60 periods still has a slightly visible dc component.

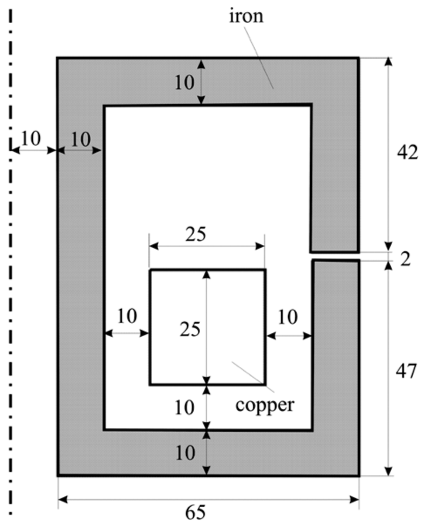

Fig. 6. Copper conductor ($\sigma = 5.7\ 10^7$ S/m) with a sinusoidal voltage of 1.4 V/m peak, 50 Hz within a conducting ferromagnetic screen ($\sigma = 1.0 \times 10^6$ S/m).

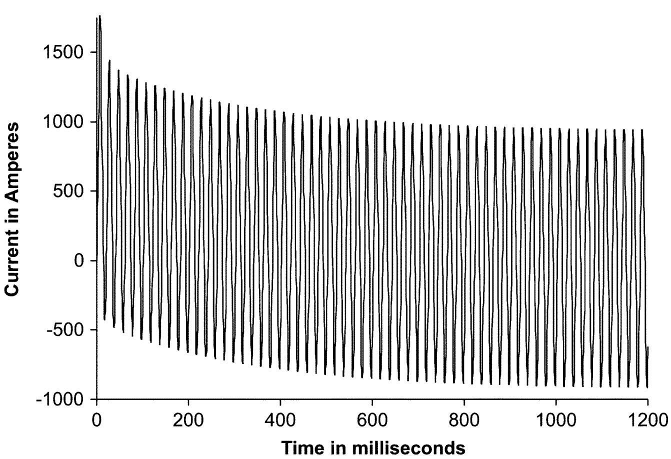

Fig. 7. Conductor current obtained by stepping through 60 periods.

V. CONCLUSION

The computational effort of the method introduced is comparable to that needed for time-stepping through one period. The necessity of using complex arithmetics is offset by the fact that the system matrix of the fixed point method is the same at each nonlinear iteration step. If the linear equations are solved by a direct method (as usual in 2-D problems), the decomposition has to be carried out just once. If iterative techniques are employed (as in 3-D problems), the equations in (4) can be solved parallel for each time step.

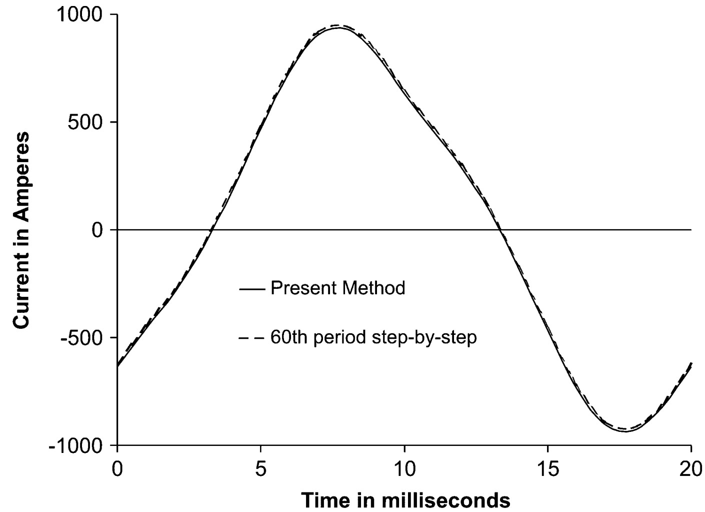

Fig. 8. Comparison of conductor current by present method and by step-by-step method.

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