On the Nonlinear Eddy Current Field Coupled to the Nonlinear Heat Transfer

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Abstract – **A semianalytical algorithm including the Fourier transform, the quasi-Newton iteration and Galerkin's finite element method, has been developed to solve the eddy current field problem and the coupled heat transfer problem in the presence of a metallic slab. The field-dependence of the magnetic permeability and the temperature-dependence of the electric and thermal conductivities of the heated slab are considered. The results obtained are reasonable and consistent with the real operation conditions in the industrial processing.**

I. INTRODUCTION

Metal parts are often hardened by high frequency induction heating where the frequency is about some hundreds of kHz. The governing equation of the concerned eddy current field problem is [1], [2], [3]

$$\frac{\partial^2 \boldsymbol{B}}{\partial x^2} + \frac{\partial^2 \boldsymbol{B}}{\partial y^2} - \mu(\sigma + \varepsilon \frac{\partial}{\partial t}) \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0} \tag{1}$$

where $\boldsymbol{B}$ is the magnetic flux density, $\mu = \mu_r \mu_0$ is the magnetic permeability, ε is the permittivity and σ is the electric conductivity. The governing equation of the heat transfer reads

$$-\nabla \cdot \kappa \nabla T + c_p \rho \frac{\partial T}{\partial t} = q \tag{2}$$

where T is the temperature, κ is the thermal conductivity, c_p is the thermal capacity, ρ is the density and q accounts for the heat source which results from the eddy current loss in the heated slab.

This paper considers the field-dependence of the magnetic permeability and the temperature-dependence of the electric and the thermal conductivities of a heated slab. To avoid the discretization in dealing with a very thin penetration depth and an open boundary, the nonlinear eddy current field is calculated by the Fourier transform with a precondition and the quasi-Newton iteration.

The temperature-dependence of $\kappa(T)$ and $\sigma(T)$, and the nonlinear magnetic reluctivity $\nu = 1/\mu$ are approximated by piecewise linear functions.

II. ALGORITHM

1. The quasi-stationary eddy current field is assumed as in the steady state and therefore described by phasors. For the 2-D rectangular coordinate case, (1) can be written as

$$\frac{\partial^2 B_x}{\partial x^2} + \frac{\partial^2 B_x}{\partial y^2} - \mathrm{j}\omega\mu(\sigma + \mathrm{j}\omega\varepsilon) B_x = 0 \,, \tag{3}$$

$$\frac{\partial^2 B_y}{\partial x^2} + \frac{\partial^2 B_y}{\partial y^2} - \mathrm{j}\omega\mu(\sigma + \mathrm{j}\omega\varepsilon) B_y = 0 \,. \tag{4}$$

The arising boundary value problem can be solved by separation of the variables. The space may be divided into three regions I, II and III according to the electromagnetic property of the medium shown in Fig. 1. Because the length of the slab is infinitely long, the Fourier transform (FT) is applied to the 2-D boundary value problem.

The Fourier transforms of B_x and B_y are [4]

$$\tilde{B}_x = \int_{-\infty}^{\infty} B_x e^{-\mathrm{j} k_x x} \, \mathrm{d}x \,, \tag{5}$$

$$\tilde{B}_y = \int_{-\infty}^{\infty} B_y e^{-\mathrm{j} k_x x} \, \mathrm{d}x \,. \tag{6}$$

The FT of (3) and (4) can then be written as

$$-k_x^2 \tilde{B}_x + \frac{\partial^2 \tilde{B}_x}{\partial y^2} - \mathrm{j}\omega\mu(\sigma + \mathrm{j}\omega\varepsilon)\tilde{B}_x = 0 \,,$$

$$-k_x^2 \tilde{B}_y + \frac{\partial^2 \tilde{B}_y}{\partial y^2} - \mathrm{j}\omega\mu(\sigma + \mathrm{j}\omega\varepsilon)\tilde{B}_y = 0 \,.$$

The solutions of the above equations provide together with the incident field the total field as

$$\tilde{B}_x = b_x \mathrm{e}^{\alpha y} + c_x \mathrm{e}^{-\alpha y} + \tilde{B}_x^{in} \,, \tag{7}$$

$$\tilde{B}_y = b_y \mathrm{e}^{\alpha y} + c_y \mathrm{e}^{-\alpha y} + \tilde{B}_y^{in} \tag{8}$$

where

$$\alpha^2 = k_x^2 + j\omega\mu(\sigma + j\omega\varepsilon)$$

and $\widetilde{B}_x^{in}$ and $\widetilde{B}_y^{in}$ are the transformed incident field components. Substituting (7) and (8) into the Fourier transformed Maxwell equation $\nabla \cdot \boldsymbol{B} = 0$, it is found that the relations

$$b_y = -\frac{jk_x b_x}{\alpha} \quad \text{and} \quad c_y = \frac{jk_x c_x}{\alpha}$$

between b_y, b_x, c_y and c_x must hold. From the geometry shown in Fig. 1, the boundary condition equations can be established to determine the integration constants.

According to Maxwell's second equation, it is known that in the 2-D case

$$E_z = \frac{1}{\mu(\sigma + j\omega\varepsilon)}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right). \tag{9}$$

Substituting (5) and (6) into the Fourier transformed equations (9), $\widetilde{E}_z$ can be obtained

$$\widetilde{E}_z = \widetilde{E}_z^{in} + \frac{\omega}{k_x}\widetilde{B}_y \tag{10}$$

where

$$\widetilde{E}_z^{in} = \frac{1}{\mu\sigma}\left(jk_x\widetilde{B}_y^{in} - \frac{\partial \widetilde{B}_x^{n}}{\partial y}\right). \tag{11}$$

It is convenient to determine the transforms of the incident field components and those of the scattered field by means of the discrete Fourier transform (DFT). By applying the inverse discrete Fourier transform (IDFT), the rms of the electric field strength can be obtained. The formula for the power density P which is the source of the heat transfer of the induction heating in the slab reads

$$P = \sigma E_z^2 \tag{12}$$

where E_z is the rms.

2. To take into account the temperature-dependence of the electric conductivity, the average temperature of the slab is taken to calculate the electric conductivity of the slab according to the piecewise-linear characteristic curve of the conductivity via the temperature. At $x = 0$, $y = d$ there is the maximum magnetic flux density from which the magnetic reluctivity is computed as follows:

- Precondition: According to the given magnetic permeability values the corresponding magnetic flux densities are calculated by means of DFT and IDFT to obtain the load line which is expressed as

$$\nu = \frac{a_1}{B} \tag{13}$$

where B is the the magnetic flux density rms for $x = 0$, $y = d$, $\nu = 1/\mu$ and a_1 is piecewise-constant. The piecewise-linear curve of the magnetic reluctivity via the magnetic flux density rms is described as

$$\nu = a_2 + a_3 B \tag{14}$$

where a_2 and a_3 are piecewise-constants.

- Quasi-Newton iteration [2] is applied to search the intersection of (13) and (14).

The advantages of the quasi-Newton method are its simple programming and its high speed of convergence, for instance, after less than 10 iterations the solution error is less than 5% as computation experiments have shown.

3. Heat transfer

Because the DFT limits the length of the slab, the heat transfer of the slab can be calculated by means of the FEM. The backward Euler scheme for the heat transfer (2) reads

$$\frac{c_p\rho(T^{n+1} - T^n)}{\Delta t} - \nabla \cdot \kappa\nabla T^{n+1} = q\,. \tag{15}$$

According to Galerkin's FEM, (15) is multiplied by the shape function Ψ and integrated by parts:

$$-\int_{\partial\Omega}\kappa\frac{\partial T^{n+1}}{\partial n}\Psi\,ds + \int_{\Omega}\left[\kappa\frac{\partial T^{n+1}}{\partial x}\frac{\partial\Psi}{\partial x} + \kappa\frac{\partial T^{n+1}}{\partial y}\frac{\partial\Psi}{\partial y}\right]d\Omega +$$

$$+\int_{\Omega}c_p\rho\frac{T^{n+1} - T^n}{\Delta t}\,d\Omega = \int_{\Omega}q\,\Psi\,d\Omega \tag{16}$$

where Ω is the field domain, Ψ is a linear shape function, $\partial\Omega$ is the boundary of the field domain Ω and $\boldsymbol{n}$ is the unit vector outward normal to the boundary.

The finite element algebra equation corresponding to (16) can be expressed by means of the matrix equation

$$[S_1][T^{n+1}] + c_p\rho[S_2]\frac{([T^{n+1}] - [T^n])}{\Delta t} = [Q] \tag{17}$$

where $[S_1]$ and $[S_2]$ are stiffness matrices, $[T^n]$ is the vector of the node temperatures for the n-th time step and $[Q]$ is a vector whose entries are the average power densities of the three nodes of the triangular elements used for the discretization.

The time step of the transient calculation is evaluated from the temperature-dependence of the thermal conductivity. The thermal conductivity can be expressed as

$$\kappa = a + bT \tag{18}$$

where a and b are piecewise-constant. According to the permitted error of κ due to the change of the temperature a vector $[\Delta T]$ can be given. In (17) $[T^n]$ is known, set $[T^{n+1}]$ according to

$$[T^{n+1}] \leqq [T^n] + [\Delta T]\,,$$

then the matrix $[S_1]$ influenced by the thermal conductivity can be calculated. For the i-th equation of (17) a value of Δt_i can be derived. Let the vectors

$$[U] = [Q] - [S_1][T^{n+1}],$$

$$[V] = c_p \rho [S_2]([T^{n+1}] - [T^n])$$

and

$$\Delta t_i = \frac{v_i}{u_i}$$

where u_i and v_i are the i-th entries of $[U]$ and $[V]$, respectively. Taking the average value of them yields an upper bound for the time step Δt:

$$h_c = \frac{1}{n_r} \sum_{i=1}^{n_r} \Delta t_i$$

where n_r is the number of the elements. The time step can be taken equal to or less than h_c. Thus the time step can be estimated, provided that the thermal conductivity has no evident change in this time step, that means the transient can be calculated step by step without iteration.

4. Iteration loop for the coupled fields

(i) The initial condition $T(x, y, 0) = T_0$ is given, calculate the loss of the eddy current field.

(ii) By the loss, the new temperature can be computed for a time step.

(iii) Taking the average value of the new temperature as the temperature of the slab, the loss of the slab can be calculated again.

(iv) Calculate a new time step, if $t < t_{max}$ go to (ii).

III. EXAMPLE

An infinitely long steel slab is heated by an inductor (Fig. 1). The boundary conditions for the 2-D heat transfer are shown in Fig. 2. In order to calculate the eddy current field by the DFT, the origin of the coordinate system is chosen at the center of the inductor. To calculate the heat transfer of the slab, the y-axis is moved to the left end of the slab. The eddy current field problem is solved in the whole space by applying the DFT, while the heat transfer is evaluated in the heated slab.

The temperature responses at $x = 0.5l$ and $x = l$ of the surface of the slab are shown in Fig. 3. The scattered electric field strength has an odd symmetry with respect to the x-axis of Fig. 1 shown in (10). In the middle of the slab the temperature increases slowly. This phenomenon is shown in Figs. 3 and 4 where the temperatures at the left and right ends have the highest values. In Fig. 4 the temperature difference between the surface and the center of the slab is about 15° C. The difference between the highest value and the lowest value along the x-axis is about 22° C.

Based on the calculations of the magnetic flux density in the whole space, the equivalent circuit inductance has been computed, and the resonance capacitance of the oscillator has been determined. The calculated inductance L is $113.35\,\mu H$ and the required capacitance C is $22.37\,nF$, which is consistent with the capacitance of a real oscillator operating in our industrial application, and shows the validity of the proposed model and the algorithm.

IV. CONCLUSION

(1) A semianalytical algorithm for the solution of the eddy current field problem coupled to the heat transfer problem is presented. The advantage is that all the formulas for the field variables are analytically available and that there is no difficulty with the discretization due to the very thin penetration depth and the open boundary.

(2) The nonlinear heat transfer is computed by Galerkin's FEM together with the backward Euler scheme where a criterion of the time step is suggested.

(3) The results show the temperature response at the surface and the local distribution of the temperature along the x-axis. Because the scattered electric field strength is of odd symmetry to the x-axis shown in (10), both the eddy current density and the temperature are lower in the middle part of the slab. In the y-direction the temperature at the surface is higher than the temperature at the center of the slab. The difference between the temperature on the surface and the temperature at the center of the slab is determined by the cooling condition of the slab.

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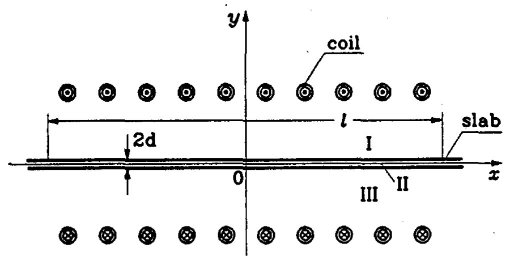

Fig.1 Inductor and slab

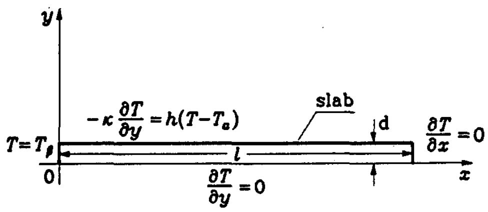

Fig.2 Boundary conditions of the heat transfer

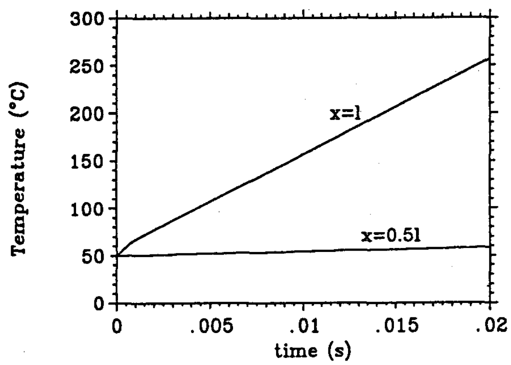

Fig. 3 Temperature response at the surface

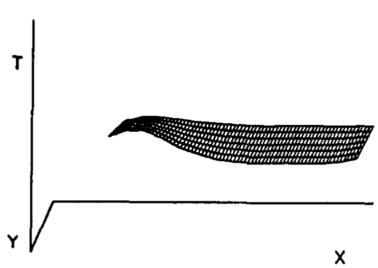

Fig. 4: Local distribution of the temperature $t = .001$ s, $f = 100$ kHz

BIOGRAPHY

Lian Gong was born in 1932, Beijing, China. She graduated from the Department of Electrical Engineering, Tsinghua University, Beijing, China in 1955. From 1955 to 1988 she was a Teaching Assistant, a Senior Lecturer and an Associate Professor, and since 1989 she has been a Professor of the Department of Electrical Engineering, Tsinghua University, Beijing, China. She is co-author of *Principle of Electromagnetic Fields* (in Chinese, Beijing, China, 1988). She is a member of the Chinese Institute of Electrical Engineering. Since 1990 she has been a visiting research professor at the Lehrstuhl für Allgemeine und Theoretische Elektrotechnik, University of Erlangen-Nürnberg.
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Reinhold Hagel (S'91) was born in Staffelstein, Fed. Rep. of Germany, 1956. He received the Dipl.-Ing. degree in electrical engineering from the University of Erlangen-Nürnberg, Erlangen, Germany, in 1983.
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Rolf Unbehauen (M'61-SM'82-FM'91) received the diploma in mathematics, the Dr. degree in Electrical Engineering and the qualification for inauguration as academic lecture from Stuttgart University, Stuttgart, Fed. Rep. of Germany, in 1954, 1957 and 1964, respectively.
From 1955 to 1966 he was a member of the Institute of Mathematics, the Computer Center and the Institute of Electrical Engineering at Stuttgart University where he was appointed Wissenschaftlicher Rat (Associate Professor) in 1965. Since 1966 he has been Full Professor of Electrical Engineering at the University of Erlangen-Nürnberg, Erlangen, Fed. Rep. of Germany.
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