# An Efficient Harmonic Method for Solving Nonlinear Time-Periodic Eddy-Current Problems

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**An iterative solution to nonlinear problems of time-periodic eddy currents is performed by directly using the time-harmonic content of the field quantities instead of time-domain techniques employing successive time steps. A linear sinusoidal steady-state field problem is solved to determine the magnetization harmonics at each iteration, with the harmonic values corrected in terms of the actual magnetic induction by applying a fixed-point procedure. To further improve its efficiency, the solution process can be started by retaining a small number of harmonics, with more harmonics subsequently added as needed to achieve the desired accuracy. The proposed method always yields stable results, even when the characteristic $\boldsymbol{B}-\boldsymbol{H}$ is strongly nonlinear, and has a superior computational efficiency with respect to various time-stepping techniques and to the "harmonic balance method."**

***Index Terms*—Eddy currents, nonlinear periodic fields, polarization fixed-point method.**

## I. Introduction

THE nonlinear time-periodic eddy-current problems are usually solved by pseudo-linear procedures where the nonlinear relationship $\boldsymbol{B}-\boldsymbol{H}$ is linearized and the material permeability is corrected in terms of the magnetic induction $\boldsymbol{B}$, based on various criteria [1]. Results produced by following this approach could be unsatisfactory in case of strong nonlinearities, with the convergence of the computational process not always insured. On the other hand, a straightforward stepping-on-in-time transient analysis follows the actual nonlinear relationship $\boldsymbol{B}-\boldsymbol{H}$, but the necessary computation time to reach the periodic steady state could be prohibitive. An analysis based on expanding the unknown field quantities in Fourier series as proposed in [2] yields large systems of nonlinear algebraic equations whose solution requires a huge computational effort. An interesting method has just been proposed [3] where a particular form of Fourier decomposition is used with the time-stepping performed through only one period. The number of systems of algebraic equations to be solved at each iteration is equal with the number of time steps.

In this paper, we present a new method for a stable and much more efficient solution of nonlinear periodic eddy-current problems. The nonlinearity is treated iteratively by the polarization fixed-point method [4]. Using a Fourier decomposition of the magnetization, each harmonic of the magnetic induction is computed by solving a linear system and, then, the magnetization is corrected at each iteration. It can be proven that the iterative numerical process is always convergent. The number of linear systems of equations that is to be solved at each iteration step is only equal to the number of harmonics taken into account, which makes this method very efficient. The iterative scheme can be started with a small number of harmonics in order to further increase the efficiency.

## II. Polarization Fixed-Point Method

The nonlinear relationship $\boldsymbol{H}=\mathbf{F}(\boldsymbol{B})$ is written in the form

$$\boldsymbol{H}=\nu\boldsymbol{B}-\boldsymbol{M} \tag{1}$$

where $\nu$ is a constant and $\boldsymbol{M}$ has a nonlinear dependence of $\boldsymbol{B}$ [4]

$$\boldsymbol{M}=\nu\boldsymbol{B}-\mathbf{F}(\boldsymbol{B})\equiv\mathbf{G}(\boldsymbol{B}). \tag{2}$$

$\nu$ is chosen such that the function $\mathbf{G}$ is a contraction, i.e.,

$$\|\mathbf{G}(\boldsymbol{B}_1)-\mathbf{G}(\boldsymbol{B}_2)\|_\mu\le\lambda\|\boldsymbol{B}_1-\boldsymbol{B}_2\|_\nu \tag{3}$$

where $\mu\equiv1/\nu$, $0<\lambda<1$, and the norm is given by

$$\|\boldsymbol{U}\|_\nu=\sqrt{\frac{1}{T}\int_0^T\int_\Omega\boldsymbol{U}\cdot(\nu\boldsymbol{U})d\Omega dt} \tag{4}$$

with $T$ being the period and $\Omega$ the space region. Starting with an arbitrary $\boldsymbol{B}$, $\boldsymbol{M}$ and, then, $\boldsymbol{B}$ are updated iteratively. An advantageous feature of the proposed method consists in the fact that the constant $\mu$ can be chosen to be the permeability of free space, $\mu=\mu_0$. This allows the construction of a simple integral equation for the current density to be solved at each iteration, as shown in Section IV.

## III. Harmonic Content and Algorithm

The time-periodic $\boldsymbol{M}$ has a Fourier series expansion in the form

$$\boldsymbol{M}(t)=\sum_{n=1,3,\ldots}\left(\boldsymbol{M}_n'\sin(n\omega t)+\boldsymbol{M}_n''\cos(n\omega t)\right). \tag{5}$$

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For the numerical computation, we retain a finite number $N$ of harmonics, $\boldsymbol{M} \cong \boldsymbol{M}_a \equiv \mathbf{Y}(\boldsymbol{M})$, the approximation $\mathbf{Y}$ being nonexpansive, i.e.,

$$\|\mathbf{Y}(\boldsymbol{M}_1) - \mathbf{Y}(\boldsymbol{M}_2)\|_\mu \leq \|\boldsymbol{M}_1 - \boldsymbol{M}_2\|_\mu. \tag{6}$$

For each harmonic $n$ of the magnetization $\boldsymbol{M}_a$, we use the complex representation

$$\boldsymbol{M}_n = \boldsymbol{M}'_n + j\boldsymbol{M}''_n \tag{7}$$

and obtain the complex magnetic induction

$$\boldsymbol{B}_n = \boldsymbol{B}'_n + j\boldsymbol{B}''_n \tag{8}$$

by solving a linear system of equations, for instance, satisfied by the unknown current density (see Section IV). From $\boldsymbol{B}_n$, we obtain the time-domain value of the magnetic induction as

$$\boldsymbol{B}(t) = \sum_{n=1,3,\ldots,2N-1} \left(\boldsymbol{B}'_n \sin(n\omega t) + \boldsymbol{B}''_n \cos(n\omega t)\right) \equiv \mathbf{L}(\boldsymbol{M}_a). \tag{9}$$

It can be shown that $\mathbf{L}$ is also nonexpansive. At each step $k \geq 1$ of the proposed iterative process, we perform the chain of operations $\boldsymbol{B}^k \xrightarrow{\mathbf{G}} \boldsymbol{M}^k \xrightarrow{\mathbf{Y}} \boldsymbol{M}_a^k \xrightarrow{\mathbf{L}} \boldsymbol{B}^{k+1}$, with $\boldsymbol{B}^1$ arbitrarily chosen. Since the composition of $\mathbf{G}$, $\mathbf{Y}$, and $\mathbf{L}$ is a contraction, the iterative process is always convergent.

Instead of systems of equations corresponding to each time step in time-domain methods, in our method one has to solve only $N$ linear complex systems at each iteration. In order to further reduce the amount of computation, we start with a small number $N$ of harmonics (even with $N = 1$). Since the inequality (6) is stronger when the number of harmonics is smaller, the rate of convergence is now higher and the amount of computation required is much reduced. When an imposed accuracy is reached, we increase the number of harmonics and perform the computation of the new set of harmonics by using the approximate values of the harmonics determined in the previous set. The computational effort needed is substantially reduced with respect to that when solving for the final set of harmonics by starting with arbitrary values for all the harmonics, especially since the harmonics in the smaller sets need not be calculated very accurately (see Tables I and II). We continue this procedure until the resultant field is accurately determined.

## IV. Integral Equation Formulation

Choosing the permeability in the computational model to be everywhere the permeability $\mu_0$ of free space, we employ an integral equation which, in the case of 2-D structures, is written in the form

$$\rho J(\boldsymbol{r}) + \frac{\mu_0}{2\pi}\frac{\partial}{\partial t}\int_\Omega J(\boldsymbol{r}')\ln\frac{1}{R}dS' = -\frac{\mu_0}{2\pi}\frac{\partial}{\partial t}\int_{\Omega_0} J_0(\boldsymbol{r}')\ln\frac{1}{R}dS' - \frac{\mu_0}{2\pi}\frac{\partial}{\partial t}\int_\Omega \boldsymbol{k}\cdot(\nabla'\times\boldsymbol{M}(\boldsymbol{r}'))\ln\frac{1}{R}dS' + C_l \tag{10}$$

where $\rho$ and $J$ are, respectively, the resistivity and the current density in the conducting regions $\Omega$, $J_0$ is the given current density in the nonferromagnetic coil regions $\Omega_0$, $\boldsymbol{r}$ and $\boldsymbol{r}'$ are the position vectors of the observation and the source points, respectively, $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$, $R = |\boldsymbol{R}|$, and $\boldsymbol{k}$ are the longitudinal unit vector. $C_l$ is a constant for each disjoint conducting region $l$ and is determined by specifying its total current. To illustrate the formulation, we choose only one conducting region $\Omega$ with a zero total current, when $C_l = 0$. $\Omega$ is divided in $I$ subdomains $\omega_i$ and $\Omega_0$ in $Q$ subdomains $\omega_{0q}$. Equation (10) is discretized as

$$\rho_m S_m J_m + \frac{\partial}{\partial t}\sum_{i=1}^{I}\beta_{mi}J_i = -\frac{\partial}{\partial t}\sum_{q=1}^{Q}\beta_{0mq}J_{0q} + \frac{\partial}{\partial t}\sum_{i=1}^{I}\boldsymbol{\gamma}_{mi}\cdot\boldsymbol{M}_i, \quad m = 1,2,\ldots,I \tag{11}$$

where $\rho_m$, $S_m$, and $J_m$ are, respectively, the resistivity, the area, and the average value of the current density of the subdomain $\omega_m$, $J_{0q}$ is the imposed current density in the subdomain $\omega_{0q}$, $\boldsymbol{M}_i$ is the magnetization in $\omega_i$, and

$$\beta_{mi} = \frac{\mu_0}{2\pi}\int_{\omega_m}\int_{\omega_i}\ln\frac{1}{R}dS'_i dS_m = \frac{\mu_0}{8\pi}\oint_{\partial\omega_m}\oint_{\partial\omega_i}R^2\ln R\, d\boldsymbol{l}_m\cdot d\boldsymbol{l}'_i \tag{12}$$

$$\beta_{0mq} = \frac{\mu_0}{2\pi}\int_{\omega_m}\int_{\omega_{0q}}\ln\frac{1}{R}dS'_q dS_m = \frac{\mu_0}{8\pi}\oint_{\partial\omega_m}\oint_{\partial\omega_{0q}}R^2\ln R\, d\boldsymbol{l}_m\cdot d\boldsymbol{l}'_q \tag{13}$$

$$\boldsymbol{\gamma}_{mi} = -\frac{\mu_0}{2\pi}\int_{\omega_m}\oint_{\partial\omega_i}\ln\frac{1}{R}d\boldsymbol{l}'_i dS_m = \frac{\mu_0}{8\pi}\oint_{\partial\omega_m}\oint_{\partial\omega_i}(2\ln R - 1)(\boldsymbol{R}\cdot\boldsymbol{n}_m)dl_m d\boldsymbol{l}'_i \tag{14}$$

where $\partial\omega_i$ is the boundary of the subdomain $\omega_i$ and $\boldsymbol{n}_i$ is the outward normal unit vector on $\partial\omega_i$. The system (11) can be written for each harmonic $n$ in a matrix form as

$$\begin{pmatrix}\beta & \delta/n \\ -\delta/n & \beta\end{pmatrix}\begin{pmatrix}J'_n \\ J''_n\end{pmatrix} = -\begin{pmatrix}A'_{0n} \\ A''_{0n}\end{pmatrix} + \begin{pmatrix}A'_{Mn} \\ A''_{Mn}\end{pmatrix} \tag{15}$$

where $\beta$ is the matrix of $\beta_{mi}$, $\delta$ is a diagonal matrix with the entries $\delta_{mm} = \rho_m S_m/\omega$, $m = 1,2,\ldots,I$, $J'_n$ and $J''_n$ are the column vectors of the real and imaginary parts of the complex current density $J_n$, $A'_{0n}$, $A''_{0n}$ and $A'_{Mn}$, $A''_{Mn}$ are, respectively, the column vectors of the real and imaginary parts of the complex vector potentials integrated over the respective subdomains $\omega_m$, $A_{0n}$ due to the imposed current density and $A_{Mn}$ to the magnetization, i.e.,

$$A_{0n} = \sum_{q=1}^{Q}\beta_{0mq}J_{0n,q}, \quad A_{Mn} = \sum_{i=1}^{I}\boldsymbol{\gamma}_{mi}\cdot\boldsymbol{M}_{n,i}. \tag{16}$$

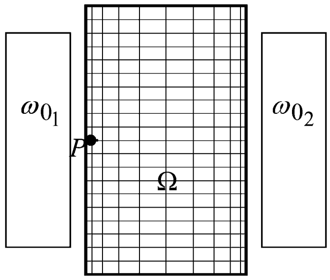


Fig. 1. Ferromagnetic conductor $\Omega$ and current-carrying coil $\Omega_0 = \omega_{0_1} \cup \omega_{0_2}$.

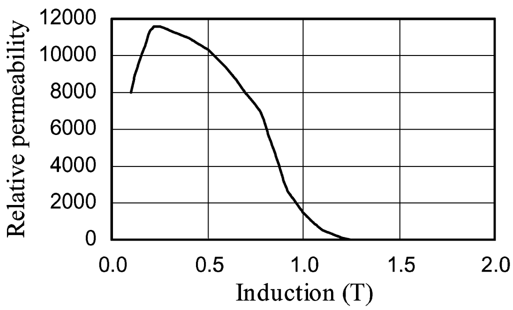


Fig. 2. Differential relative permeability.

$A_{0n}$ is the same for all iterations, while $A_{Mn}$ is to be corrected at each iteration.

After solving the system (15), the average value of the complex magnetic induction in the subdomain $\omega_m$ is computed as

$$\boldsymbol{B}_{n,m} = -\frac{1}{S_m}\left(\sum_{i=1}^{I} \boldsymbol{\gamma}_{mi} J_{n,i} + \sum_{i=1}^{I} \overline{\overline{\zeta}}_{mi} \boldsymbol{M}_{n,i}\right) + \boldsymbol{B}_{0n,m} \tag{17}$$

where $\boldsymbol{B}_{0n,m}$ is the magnetic induction due to the imposed current density, the same at all the iterations

$$\boldsymbol{B}_{0n,m} = -\frac{1}{S_m} \sum_{q=1}^{Q} \boldsymbol{\gamma}_{mq} J_{0n,q} \tag{18}$$

and

$$\overline{\overline{\zeta}}_{mi} = \frac{\mu_0}{2\pi} \oint_{\partial\omega_m} \oint_{\partial\omega_i} \ln R \left(d\boldsymbol{l}_m d\boldsymbol{l}'_i\right) \tag{19}$$

the latter being expressed in terms of the dyads $(d\boldsymbol{l}_m d\boldsymbol{l}'_i)$. The numerical approximation of $\boldsymbol{B}_{n,m}$ due to the averaging is nonexpansive and, thus, always preserves the convergence of the iterative process even in the case of large differences in the differential magnetic permeability. At any time $t$, the magnetic induction is obtained with (9), the magnetization is corrected with (2) and, then, used for computing the new complex expression in (7), with

$$\boldsymbol{M}'_n = \frac{2}{T} \int_0^T \boldsymbol{M}(t) \sin(n\omega t) dt$$
$$\boldsymbol{M}''_n = \frac{2}{T} \int_0^T \boldsymbol{M}(t) \cos(n\omega t) dt. \tag{20}$$

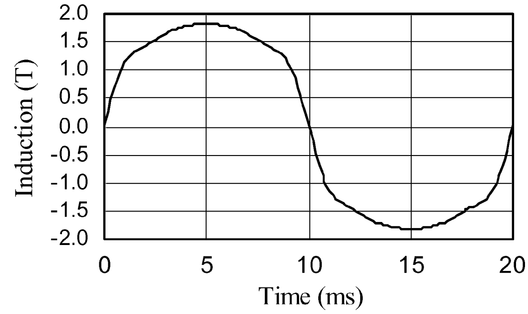


Fig. 3. Vertical component of the magnetic induction at point $P$ in Fig. 1.

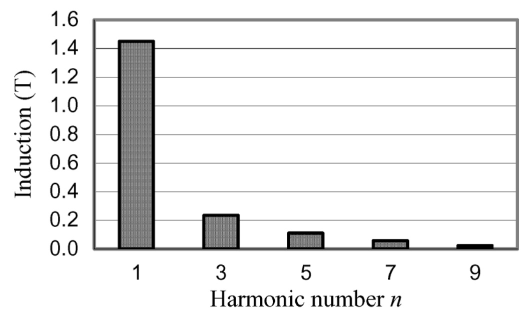


Fig. 4. Harmonic content of induction at point $P$ in Fig. 1.

## V. Computation Examples

In a first example, we consider a long, 10 × 20 mm ferromagnetic conductor $\Omega$ surrounded by a 4 × 12 mm coil with 200 turns, carrying a 50-Hz sinusoidal current of rms value of 50 A, as shown in a cross section in Fig. 1. The $\boldsymbol{B} - \boldsymbol{H}$ characteristic is strongly nonlinear, with the differential relative permeability having values between 1.3 and 11374, as specified in Fig. 2. First, we consider only one harmonic and, then, successively, the sets of the first 2, 3, 4, and 5 harmonics. The number of iterations in the polarization fixed-point method, the number of linear systems to be solved, and the relative error

$$\varepsilon_r = \frac{\|\boldsymbol{M}^k - \boldsymbol{M}^{k-1}\|_\mu}{\|\boldsymbol{M}^k\|_\mu} \tag{21}$$

are given in Table I. The time dependence of the vertical component of the magnetic induction at the point $P$ in Fig. 1 is given in Fig. 3 and its content of harmonics in rms values at the same point is plotted in Fig. 4.

In a second example, we determine the magnetic induction in a conducting ferromagnetic shield $\Omega$ of outer cross-sectional dimensions of 20 × 20 mm and thickness of 2 mm surrounding a 5 × 8 mm coil $\omega_{0_1} \cup \omega_{0_2}$ which carries a 50-Hz sinusoidal current that produces a magnetic induction at the point $P$ in Fig. 5 as plotted in Fig. 6. The magnetic material of the shield has the same permeability as in the previous example.

The harmonics considered successively, the number of iterations and of linear systems solved, and also the corresponding errors are given in Table II. For this example, the increase in accuracy as the number of iterations performed increases is indicated in Fig. 7. The peaks in this graph are due to the fact that we performed the computation with sets containing an increasing number of harmonics. After the last iteration for a given

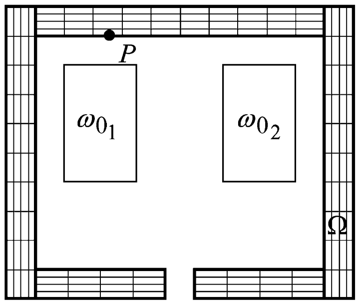


Fig. 5. Ferromagnetic shield $\Omega$ and current coil $\omega_{0_1} \cup \omega_{0_2}$.

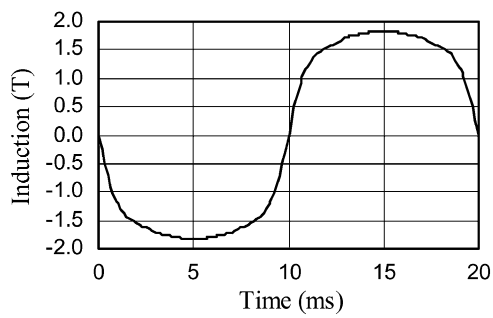


Fig. 6. Horizontal component of the magnetic induction at point $P$ in Fig. 5.

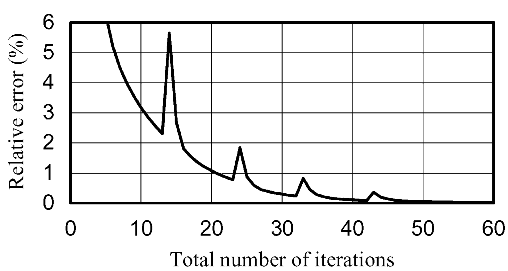


Fig. 7. Evolution of the relative error (related to Table II).

TABLE I
NUMBER OF ITERATIONS AND RELATIVE ERROR FOR VARIOUS NUMBERS OF HARMONICS

| Harmonics considered | 1 | 1,3 | 1,3,5 | 1,3,5,7 | 1,3,5,7,9 |
|---|---|---|---|---|---|
| No.of iterations | 4 | 9 | 9 | 11 | 57 |
| No.of systems | 4 | 18 | 27 | 44 | 285 |
| Relative error (%) | 5.21 | 1.75 | 0.58 | 0.18 | 0.0096 |

set, the error (21) has the value shown in Table II. Computation for a subsequent set is started with arbitrary values for the new harmonics in the set (e.g., zero values) which determines a jump in error for the first iteration in this set.

## VI. CONCLUSION

We employed a PC with a 1-GHz processor and 1-GB RAM memory and considered 120 time steps for one period in order to calculate the instantaneous values in (5) and (9) for both examples. No overrelaxation has been used in the iterative process and less then 1 min has been required for the entire computation in the more complex, second example. As shown in Table II, we had to solve only 60 linear systems to obtain an error [see (21)] $\varepsilon_r \cong 0.26\%$ in the case of three harmonics, 100 systems for $\varepsilon_r \cong 0.1\%$ when considering four harmonics and only 250 systems to reach a high accuracy, with $\varepsilon_r \cong 0.01\%$, when five harmonics are taken into account. If, instead of starting with the fundamental harmonic and then adding successively one more harmonic, the computation for the second example is performed from the beginning by taking all the first five harmonics, then a number of 64 iterations is required to obtain an error of about 0.01%, i.e., a number of 320 systems to be solved. This shows the advantage of using the proposed procedure by starting it with a smaller number of harmonics.

TABLE II
NUMBER OF ITERATIONS AND RELATIVE ERROR FOR VARIOUS NUMBERS OF HARMONICS

| Harmonics considered | 1 | 1,3 | 1,3,5 | 1,3,5,7 | 1,3,5,7,9 |
|---|---|---|---|---|---|
| No.of iterations | 13 | 10 | 9 | 10 | 30 |
| No.of systems | 13 | 20 | 27 | 40 | 150 |
| Relative error (%) | 2.56 | 0.86 | 0.26 | 0.098 | 0.0096 |

Among the methods previously presented in the literature, the time-domain technique in [3] is superior since it requires the solution of a number of systems equal to the product of the number of iterations with the number of time steps through only one period. The method proposed in the present paper requires the solution of a much smaller number of linear systems, equal to the product of the number of iterations with the number of harmonics taken into account, which shows clearly its higher efficiency.


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