On the 3-D Inhomogeneous Induction Heating of a Shell

L. Gong, R. Hagel, and R. Unbehauen
Lehrstuhl für Allgemeine und Theoretische Elektrotechnik, Universität Erlangen-Nürnberg
Cauerstr. 7, D-91058 Erlangen, Germany

Abstract—**To overcome the difficulty of discretization due to a very small penetration depth, the 3-D eddy current field problem is solved by the longitudinal component method together with a discrete Fourier transform (DFT) in cylindrical coordinates. Galerkin's FEM is applied to calculate the 3-D heat transfer in a conducting and permeable moving shell. Numerical results are presented.**

I. Introduction

A moving steel shell is heated by an eccentric inductor idealized as a circular line current as shown in Fig. 1. Because the shell moves with a low velocity along its longitudinal direction, the influence of the velocity-dependence on the constitutive relations can be neglected. The electromagnetic field due to the inductor current is considered as the incident field which will be calculated in region III (Fig. 1) by means of Biot-Savart law. The total field in region III is represented as the superposition of the incident field and the scattered field. The scattered field in region III as well as the total field in regions I and II satisfy the homogeneous Helmholtz equations. For time harmonic fields with frequency $\omega = 2\pi f$, the phasors $\boldsymbol{B}$ and $\boldsymbol{E}$ (i.e., the vectors of complex quantities associated to the sinusoidal field components in usual manner) are introduced so that the source-free Helmholtz equations read as

$$\nabla^2 \boldsymbol{B} + k^2 \boldsymbol{B} = \boldsymbol{0} \tag{1}$$

and

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = \boldsymbol{0} \tag{2}$$

with $k^2 = \omega^2 \mu_0 \varepsilon_0$ in air and $k^2 = -\mathrm{j}\,\omega\mu(\sigma + \mathrm{j}\,\omega\varepsilon_0)$ in the permeable and conducting material of the shell, where μ is the permeability, σ is the electric conductivity, ε_0 is the vacuum permittivity. To avoid the difficulties of discretization owing to a very small penetration depth and the open boundary in numerical methods, separation of variables is preferred. In cylindrical coordinates r, φ, z only the longitudinal components can be solved by separation of the variables. Since the transverse components and the longitudinal ones are related to each other, the transverse components can be expressed in terms of the longitudinal ones [1]. The governing equation of the steady heat transfer reads

$$-\nabla\cdot\kappa\nabla T + c_p \rho \boldsymbol{v}\cdot\nabla T = q \tag{3}$$

where T is the temperature, κ is the thermal conductivity, c_p is the thermal capacity, ρ is the mass density and q represents the heat source which results from the eddy current loss. To solve the steady heat transfer, a 3-D calculation is performed inside the shell and a convenient method is Galerkin's FEM [2].

II. Algorithm

A. 3-D eddy current field

The components of both the magnetic field and the electric field can be expanded into Fourier series with respect to φ. Due to the symmetry of the problem (Fig. 1), the n-th order harmonics B_{rn}, B_{zn} and $E_{\varphi n}$ are of even symmetry with respect to φ while $B_{\varphi n}$, E_{rn} and E_{zn} are of odd symmetry with respect to φ. Assuming the shell is infinitely long, we apply the Fourier transform (FT) with respect to the z-variable:

$$\widetilde{B}_{zn} = \int_{-\infty}^{\infty} B_{zn}\,\mathrm{e}^{-\mathrm{j}kz}\,\mathrm{d}z\ ,\quad \widetilde{E}_{zn} = \int_{-\infty}^{\infty} E_{zn}\,\mathrm{e}^{-\mathrm{j}kz}\,\mathrm{d}z\ . \tag{4a,b}$$

Applying the FT and the separation of the variables to the longitudinal components of (1) and (2), we obtain Bessel equations for the transformed Fourier coefficients of the longitudinal components:

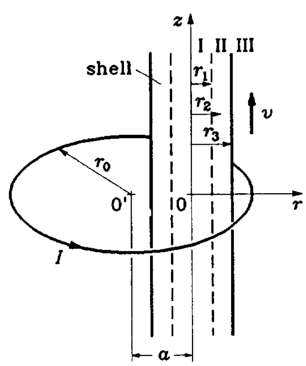

Fig.1. Shell and inductor

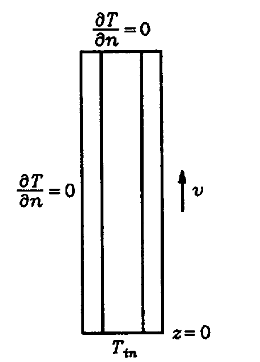

Fig.2. Boundary conditions for the heat transfer

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$$\frac{\partial^2 \widetilde{B}_{zn}}{\partial r^2} + \frac{1}{r}\frac{\partial \widetilde{B}_{zn}}{\partial r} + \left(\Lambda^2 - \frac{n^2}{r^2}\right)\widetilde{B}_{zn} = 0\,, \quad (5)$$

$$\frac{\partial^2 \widetilde{E}_{zn}}{\partial r^2} + \frac{1}{r}\frac{\partial \widetilde{E}_{zn}}{\partial r} + \left(\Lambda^2 - \frac{n^2}{r^2}\right)\widetilde{E}_{zn} = 0\,. \quad (6)$$

In regions I and III, we have $\Lambda^2 = \omega^2 \mu_0 \varepsilon_0 - k_z^2$ and in region II $\Lambda^2 = -\mathrm{j}\,\omega\mu(\sigma + \mathrm{j}\,\omega\varepsilon_0) - k_z^2$. The solutions of the above Bessel equations for the three field regions can generally be written as

$$\widetilde{B}_{zn} = C_{1n} J_n(\Lambda r) + C_{2n} H_n^{(2)}(\Lambda r)\,, \quad (7)$$

$$\widetilde{E}_{zn} = D_{1n} J_n(\Lambda r) + D_{2n} H_n^{(2)}(\Lambda r) \quad (8)$$

where J_n and $H_n^{(2)}$ are Bessel functions with integer order n. In region I, $C_{2n} = D_{2n} = 0$ and in region III, $C_{1n} = D_{1n} = 0$ must be set. Thus for every n, 8 quantities are to be evaluated. The transformed Fourier coefficients of the 4 transverse components can be expressed by $\widetilde{B}_{zn}$ and $\widetilde{E}_{zn}$ [3]. The boundary conditions based on the continuity of the 4 tangential components (Appendix) are established at $r = r_3$:

$$\frac{1}{\mu_r}\widetilde{B}_{zn}^{II}\,|_{r=r_3} = \widetilde{B}_{zn}^{III}\,|_{r=r_3} + \widetilde{B}_{zn}^{in}\,|_{r=r_3}\,, \quad (9)$$

$$\widetilde{E}_{zn}^{II}\,|_{r=r_3} = \widetilde{E}_{zn}^{III}\,|_{r=r_3}\,, \quad (10)$$

$$\frac{1}{\mu_r}\widetilde{B}_{\varphi n}^{II}\,|_{r=r_3} = \widetilde{B}_{\varphi n}^{III}\,|_{r=r_3} + \widetilde{B}_{\varphi n}^{in}\,|_{r=r_3}\,, \quad (11)$$

$$\widetilde{E}_{\varphi n}^{II}\,|_{r=r_3} = \widetilde{E}_{\varphi n}^{III}\,|_{r=r_3} + \widetilde{E}_{\varphi n}^{in}\,|_{r=r_3} \quad (12)$$

where the incident field is developed into a Fourier series and then transformed. Together with the four analog boundary conditions at $r = r_1$ the above 8 coefficients can be determined. Performing the inverse FT, finally the components E_r, E_φ and E_z of $\boldsymbol{E}$ are obtained. The electric power density in the shell is then calculated by

$$p = \sigma(E_r^2 + E_\varphi^2 + E_z^2)\,. \quad (13)$$

At $r = r_3$ and $z = 0$ (Fig.1) $|\boldsymbol{B}|$ has its maximum value and the nonlinear μ has to be adapted to this value. Therefore an initial value $\mu_r^{(0)}$ is set to calculate the eddy current field problem, the maximum value of the magnetic flux density and the corresponding $\mu_r^{(1)}$. If the relative error of μ_r is less than 5%, then go on to calculate the heat transfer problem, otherwise an under relaxation formula of iteration $\mu^{(i)} = 0.5(\mu^{(i-1)} + \mu^{(i)})$ is applied and the eddy current field has to be computed again.

B. Heat transfer

According to Galerkin's FEM we discretize the shell and choose prisms as finite elements. Therefore 3-D Cartesian coordinates are utilized instead of 3-D cylindrical coordinates. Eq. (3) is multiplied by a shape function Ψ and integrated by parts:

$$-\int_{\partial\Omega} \kappa \frac{\partial T}{\partial n}\Psi \mathrm{d}s + \int_{\Omega} [\kappa \frac{\partial T}{\partial x}\frac{\partial \Psi}{\partial x} + \kappa \frac{\partial T}{\partial y}\frac{\partial \Psi}{\partial y} + \kappa \frac{\partial T}{\partial z}\frac{\partial \Psi}{\partial z}]\,\mathrm{d}\Omega + \int_{\Omega} c_p \rho v \frac{\partial T}{\partial z}\Psi \mathrm{d}\Omega = \int_{\Omega} q\,\Psi \mathrm{d}\Omega \quad (14)$$

where $\boldsymbol{v} = \boldsymbol{e}_z v$ is the velocity of the shell. The finite element algebraic equation corresponding to (14) can be expressed as the matrix equation

$$[S][T] = [Q] \quad (15)$$

where $[T]$ is the vector of the node temperatures, $[S]$ is the stiffness matrix and $[Q]$ is a vector whose entries are the power densities of the prism elements used. Because the shell moves along the z-axis, the v-dependence makes $[S]$ non-symmetric. However, due to the low speed of the shell, (15) can be solved by Gauss elimination without numerical oscillations.

To consider the T-dependence of κ, $\kappa_i^{(0)}$, $i = 1, 2, \dots, m$, are set, where m is the number of elements of the shell. After the described computation step of the heat transfer problem, according to the distribution of the temperature obtained and the piecewise-linear property of κ, we calculate $\kappa_i^{(1)}$, $i = 1, 2, \dots, m$ and the average error $\alpha = \frac{1}{m}\sum_{i=1}^{m}\frac{\kappa_i^{(1)} - \kappa_i^{(0)}}{\kappa_i^{(1)}}$. If α is less than some permissible value, the results are printed, else the iteration procedure and the relaxation formula are recalled. According to the movement of the shell (Fig. 1), we denote the temperature of the part of the shell far in front of the inductor by T_{in} while T_{out} denotes the temperature of the part of the shell far behind the inductor. The behaviour of σ with respect to T_{out} can be expressed in piecewise-linear form as

$$\sigma = \sigma_0(1 - c_1 \Delta T) \quad (16)$$

where σ_0 is the electric conductivity at the ambient temperature T_0, c_1 is a constant and $\Delta T = T_{out} - T_0$. In the same time, the increase of σ leads to a rise of T_{out} that may be approximately described as a linear function:

$$T_{out} = c_2 + c_3\,\sigma\,. \quad (17)$$

The two constants c_2 and c_3 are determined by choosing two values σ_1 and σ_2. The above nonlinear eddy current problem and the nonlinear heat transfer problem are solved for σ_1 and σ_2 to obtain the two corresponding T_{out1} and T_{out2}. From (σ_1, T_{out1}) and (σ_2, T_{out2}) one gets c_2 and c_3 from (17). In this case, σ can be solved from the two linear simultaneous equations (16) and (17) to get T_{out}.

III. EXAMPLE

A conducting and permeable shell heated by an inductor (Fig. 1) where $I = 200$ A, $f = 100$ kHz, the radius of the inductor $r_0 = 10$ mm, the inner radius of the shell $r_1 = 4$ mm while the outer radius is $r_3 = 6$ mm and $v = 1$ m/s, has been considered. For the calculation, only $n = 0$ and $n = 1$ of the Fourier series are taken since the error of neglecting the higher harmonics is less than 5%. The natural boundary conditions for the 3-D heat transfer problem with $c_p = 464$ J/kg·K, $\rho = 7800$ kg/m^3 and $T_{in} = 20\,^\circ$C are shown in Fig. 2. In order to solve the eddy current field problem, the origin of the coordinate system is chosen at the center of the shell in the plane of the inductor where the eccentric distance is $a = 3.5$ mm (Fig. 1). To solve the heat transfer problem in the case of a moving shell, the lower end of the shell is placed at $z = 0$ (Fig. 2).

The piecewise-linear descriptions used are: $\mu_r = 100$, $B \leqq 0.2\,$T; $\mu_r = 120 - c_m B$, $c_m = 100/$T, $0.2\,T < B \leqq 1.0\,$T; $\mu_r = 20$, $B > 1.0\,$T, where B is the rms of $\boldsymbol{B}$. $\sigma_0 = 5 \times 10^6\,$S/m, $c_1 = 0.0021/^\circ$C, $T_0 = 20\,^\circ$C, $T < 200\,^\circ$C and $\kappa = 55.4(1 - c_4 T)\,$W/K·m, $c_4 = 0.0235/^\circ$C, $T < 200\,^\circ$C.

From the distribution of the magnitude of $\boldsymbol{B}$ along the z-axis shown in Fig. 3, the skin effect is evident. The distribution of the magnitude of $\boldsymbol{E}$ along the z-axis decays as shown in Fig. 4. The distributions of the temperature along the z-axis and in the φ-direction are shown in Figs. 5 and 6, respectively. Obviously, an inhomogeneous induction heating is achieved.

IV. CONCLUSIONS

1. A semi-analytical method for computation of the 3-D eddy current field coupled to the 3-D heat transfer is presented. The advantage is that all the formulas for the field variables are analytically available and there is no difficulty with the discretization.
2. The 3-D steady heat transfer problem is computed by Galerkin's FEM considering the movement of the heated shell whose speed is less than 10 m/s.
3. The results show that with the eccentric inductor inhomogeneous induction heating is clearly achieved.

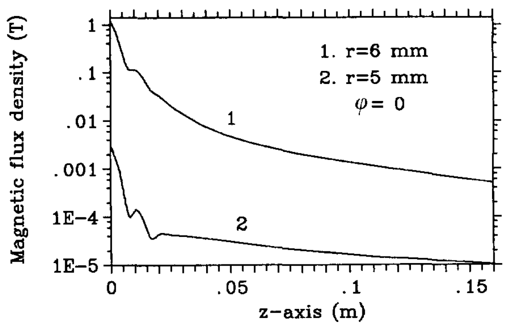

Fig. 3. Distribution of the magnitude of the magnetic flux density against z

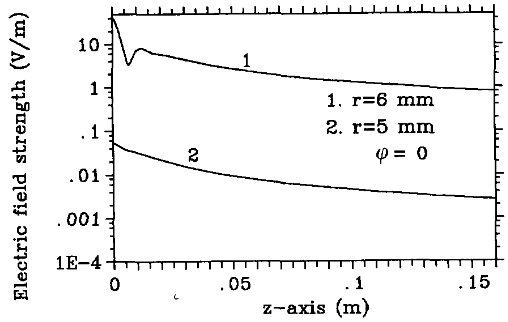

Fig. 4. Distribution of the magnitude of the electric field strength against z

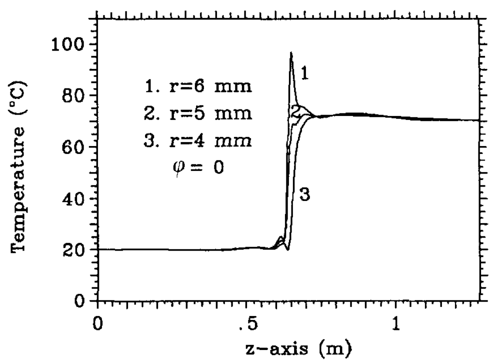

Fig. 5. Distribution of the temperature against z

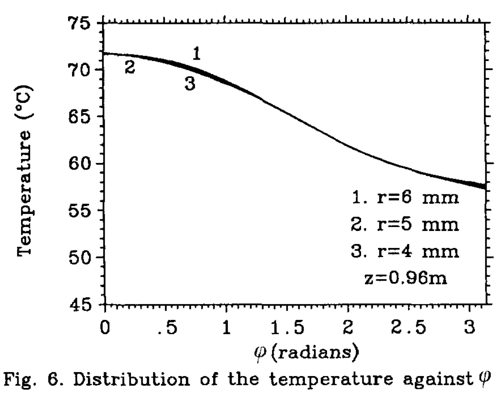

Fig. 6. Distribution of the temperature against φ

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Appendix Uniqueness Theorem for the Boundary Value Problem of the Helmholtz Equation in a Closed Region Divided into a Number of Subregions [4]

The region G is surrounded by a closed surface S, n is the number of subregions G_i, V_i $(i = 1, 2, \ldots, n)$ and S_i $(i = 1, 2, \ldots, n)$ denote the volume and closed surface of each subregion, respectively. S_i consists of a part of the boundary S denoted as S_{i0} and the interfaces of neighbour subregions i and j denoted as S_{ij}. m is the number of interfaces in region G. The medium of each subregion is uniform and the corresponding complex constitutive parameters are $\dot{\varepsilon}_i = \varepsilon_i' - \mathrm{j}\,\varepsilon_i''$, $\dot{\mu}_i = \mu_i' - \mathrm{j}\,\mu_i''$ and σ_i.

Statement of the uniqueness theorem:

For a time harmonic field, if the source current densities $\boldsymbol{J}_i$ and the source magnetic current density $\boldsymbol{J}_{mi}$ are given everywhere in V_i $(i = 1, 2, \ldots, n)$, $\boldsymbol{n} \times \boldsymbol{E} \mid_{S_{i0}}$ or $\boldsymbol{n} \times \boldsymbol{H} \mid_{S_{i0}}$ are given everywhere at S_{i0}, and $\boldsymbol{n} \times \boldsymbol{E} \mid_{S_{ij}}$ and $\boldsymbol{n} \times \boldsymbol{H} \mid_{S_{ij}}$ are continuous everywhere at S_{ij}, then the solutions $\boldsymbol{E}_i$ and $\boldsymbol{H}_i$ of Maxwell's equations or Helmholtz equations are unique.

Proof: Assume $\boldsymbol{E}_{i1}$, $\boldsymbol{H}_{i1}$ and $\boldsymbol{E}_{i2}$, $\boldsymbol{H}_{i2}$ $(i = 1, 2, \ldots, n)$ are two sets of solutions. Each set must satisfy Maxwell's equations and the conditions imposed at the boundary and interfaces. According to the superposition theorem, the difference field

$$\Delta\boldsymbol{E}_i = \boldsymbol{E}_{i1} - \boldsymbol{E}_{i2}\,,\quad \Delta\boldsymbol{H}_i = \boldsymbol{H}_{i1} - \boldsymbol{H}_{i2}\,,$$

must satisfy Poynting's theorem as derived from Maxwell's equations in V_i, providing

$$\oint_{S_i} (\Delta\boldsymbol{E}_i \times \Delta\boldsymbol{H}_i^*) \cdot \boldsymbol{n}_i \,\mathrm{d}s = -\int_{V_i} (\mathrm{j}\omega\dot{\varepsilon}_i)^* \mid \Delta\boldsymbol{E}_i \mid^2 \mathrm{d}v - \int_{V_i} \mathrm{j}\,\omega\dot{\mu}_i \mid \Delta\boldsymbol{H}_i \mid^2 \mathrm{d}v - \int_{V_i} \sigma_i \mid \Delta\boldsymbol{E}_i \mid^2 \mathrm{d}v\,. \qquad \text{(A-1)}$$

Due to the sources being given everywhere in V_i, $\Delta\boldsymbol{J}_i = \boldsymbol{0}$, $\Delta\boldsymbol{J}_{mi} = \boldsymbol{0}$, so that no term $\Delta\boldsymbol{J}_i$ or $\Delta\boldsymbol{J}_{mi}$ occurs on the right-hand side of (A-1). There are two terms at S_{lk} of the neighbour subregions l and k to be considered in the sum of the integrals with respect to the m interfaces. Both the two normal unit vectors $\boldsymbol{n}_{lk}$ and $\boldsymbol{n}_{kl}$ for neighbour subregions l and k, respectively, point outwards their own subregion, $\boldsymbol{n}_{lk} = -\boldsymbol{n}_{kl}$. Thus the left-hand side of (A-1) is

$$\sum_{i=1}^{n} \oint_{S_i} (\Delta\boldsymbol{E}_i \times \Delta\boldsymbol{H}_i^*) \cdot \boldsymbol{n}_i \,\mathrm{d}s = \sum_{i=1}^{n} \int_{S_{i0}} (\Delta\boldsymbol{E}_i \times \Delta\boldsymbol{H}_i^*) \cdot \boldsymbol{n}_{i0} \,\mathrm{d}s + \sum_{1}^{m} \int_{S_{lk}} (\Delta\boldsymbol{E}_l \times \Delta\boldsymbol{H}_l^* - \Delta\boldsymbol{E}_k \times \Delta\boldsymbol{H}_k^*) \cdot \boldsymbol{n}_{lk} \,\mathrm{d}s\,.$$

Because the tangential components of $\boldsymbol{E}_i$ or $\boldsymbol{H}_i$ are given everywhere at the outer boundary S_{i0},

$$\boldsymbol{n} \times \Delta\boldsymbol{E}_i \mid_{S_{i0}} = \boldsymbol{0} \quad \text{or} \quad \boldsymbol{n} \times \Delta\boldsymbol{H}_i \mid_{S_{i0}} = \boldsymbol{0}\,. \qquad \text{(A-2)}$$

According to the continuity of the tangential components of the electric and magnetic fields everywhere on the interface S_{lk}, we have

$$\boldsymbol{n} \times \Delta\boldsymbol{E}_l \mid_{S_{lk}} = \boldsymbol{n} \times \Delta\boldsymbol{E}_k \mid_{S_{lk}}$$

and

$$\boldsymbol{n} \times \Delta\boldsymbol{H}_l \mid_{S_{lk}} = \boldsymbol{n} \times \Delta\boldsymbol{H}_k \mid_{S_{lk}}\,. \qquad \text{(A-3)}$$

Eqns. (A-2) and (A-3) yield zero for the left-hand side of (A-1), so (A-1) becomes

$$-\sum_{i=1}^{n} \int_{V_i} (\mathrm{j}\omega\dot{\varepsilon}_i)^* \mid \Delta\boldsymbol{E}_i \mid^2 \mathrm{d}v - \sum_{i=1}^{n} \int_{V_i} \mathrm{j}\,\omega\dot{\mu}_i \mid \Delta\boldsymbol{H}_i \mid^2 \mathrm{d}v - \sum_{i=1}^{n} \int_{V_i} \sigma_i \mid \Delta\boldsymbol{E}_i \mid^2 \mathrm{d}v = 0\,. \qquad \text{(A-4)}$$

The real and imaginary parts in (A-4) must be equal to zero individually, i.e.,

$$\sum_{i=1}^{n} \int_{V_i} \omega\varepsilon_i'' \mid \Delta\boldsymbol{E}_i \mid^2 \mathrm{d}v + \sum_{i=1}^{n} \int_{V_i} \omega\mu_i'' \mid \Delta\boldsymbol{H}_i \mid^2 \mathrm{d}v + \sum_{i=1}^{n} \int_{V_i} \sigma_i \mid \Delta\boldsymbol{E}_i \mid^2 \mathrm{d}v = 0\,, \qquad \text{(A-5)}$$

$$\sum_{i=1}^{n} \int_{V_i} \omega\varepsilon_i' \mid \Delta\boldsymbol{E}_i \mid^2 \mathrm{d}v - \sum_{i=1}^{n} \int_{V_i} \omega\mu_i' \mid \Delta\boldsymbol{H}_i \mid^2 \mathrm{d}v = 0\,. \qquad \text{(A-6)}$$

If any one of ε_i'', μ_i'' or σ_i is not zero then $\mid \Delta\boldsymbol{E}_i \mid^2 = 0$, $\mid \Delta\boldsymbol{H}_i \mid^2 = 0$, or $\Delta\boldsymbol{E}_i = 0$, $\Delta\boldsymbol{H}_i = 0$. The uniqueness of the subregion solutions is proved.