An Efficient Harmonic Balance Method for Nonlinear Eddy-Current Problems

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A method is presented which determines the steady-state solution of nonlinear eddy current problems. The unknown potentials are represented by Fourier-series and the nonlinear behavior of the material is split into a linear and a nonlinear term using a fixed-point technique. This approach leads to decoupled linear equations for each harmonic component. To take the nonlinearity into account, several fixed-point iterations have to be made. The method avoids calculating transient processes which normally have to be stepped through if using time-stepping methods. The present method is illustrated by two 2-D examples.

Index Terms—**Eddy currents, nonlinear equations, nonlinear magnetics, periodic functions.**

I. Introduction

In nonlinear dynamic electromagnetic problems with periodic excitation the steady-state solution is often sufficient. With time-stepping methods a lot of periods may have to be stepped through until steady state is achieved. Such strategies like the so-called "brute force" method (see, e.g., [4]) are, therefore, computationally expensive.

For nonlinear static problems on the other hand, the fixed-point method is a promising technique [1]. The fixed-point method means that the nonlinear relation of the material is split into a linear and a nonlinear term. The nonlinear term is then updated from the previous solution in each iteration step. This method can also be used for the time domain solution of nonlinear dynamic problems with sinusoidal excitation [2]. The approach in [2] leads to a number of linear equations, where the equations are decoupled for each time instant within a period. Several iteration steps have to be made to take the nonlinear behavior of the material into account.

Another way for obtaining steady state is by introducing Fourier-series leading to what is known as the harmonic balance method. In [3] and [5], Fourier-series are used for the magnetic vector potential $\mathbf{A}$, the magnetic reluctivity ν, and the excitation $\mathbf{J}$. Here, the amplitude of a harmonic component depend on all amplitudes obtained at the previous iteration step.

In this paper, the harmonic balance method and the fixed-point method are combined. By using the fixed-point method, a Fourier-series for the reluctivity ν is not needed and the element matrices of the equation system became constant. The nonlinear term appears on the right-hand side only and, therefore, only this side had to be updated in each iteration step. The excitation as well as the unknowns are represented by Fourier-series with a finite number of harmonics. This leads to a number of linear equation systems, which are decoupled for each harmonic component.

II. Method

For eddy current problems with nonlinear materials, the differential equation has the general form

$$\mathbf{S}(\mathbf{x}) \cdot \mathbf{x}(t) + \frac{d}{dt}\left(\mathbf{M}(\mathbf{x}) \cdot \mathbf{x}(t)\right) = \mathbf{f}(t). \tag{1}$$

The matrices $\mathbf{S}$ and $\mathbf{M}$ in (1) depend on the solution $\mathbf{x}(t)$ and must, therefore, be updated in each iteration step. If the formulation leading to (1) is based on the flux density $\mathbf{B}$, the magnetic field intensity $\mathbf{H}$ can be separated in a linear and nonlinear term [1]

$$\mathbf{H}(\mathbf{B}) = \nu_{\mathrm{FP}}\mathbf{B} - \mathbf{M}_{\mathrm{FP}}(\mathbf{B}). \tag{2}$$

Here, $\mathbf{M}_{\mathrm{FP}}$ is a magnetization-like quantity which includes the nonlinear behavior of the material and ν_{FP} is a fixed value which influences the convergence of the method.

In eddy current regions, the magnetic vector potential $\mathbf{A}$ and the time integrated electric scalar potential v can be introduced as

$$\mathbf{B} = \operatorname{curl} \mathbf{A} \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad}\frac{\partial \mathrm{v}}{\partial t} \tag{3}$$

where $\mathbf{E}$ is the electric field intensity. With these potentials, using (2), Ampere's law and the law of charge conservation can be written as

$$\begin{aligned} \nu_{\mathrm{FP}}\operatorname{curl}(\operatorname{curl}\mathbf{A}) + \sigma\left(\frac{\partial \mathbf{A}}{\partial t} + \operatorname{grad}\frac{\partial \mathrm{v}}{\partial t}\right) &= \operatorname{curl}\mathbf{M}_{\mathrm{FP}} \\ \operatorname{div}\left(\sigma\frac{\partial \mathbf{A}}{\partial t} + \sigma\operatorname{grad}\frac{\partial \mathrm{v}}{\partial t}\right) &= 0. \end{aligned} \tag{4}$$

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If Galerkin's method is applied to (4), an algebraic equation system is obtained

$$\mathbf{S}\cdot\mathbf{x}(t)+\mathbf{M}\cdot\dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x},t) \tag{5}$$

where the vector $\mathbf{x}(t)$ contains the unknown time dependent potentials $\mathbf{A}$ and v and $\mathbf{f}(\mathbf{x},t)$ contains the excitation and a term which corresponds to $\mathbf{M}_{\mathrm{FP}}$. Since $\mathbf{M}_{\mathrm{FP}}$ depends on $\mathbf{B}$, the right-hand side of (5) depends on $\mathbf{x}$.

Due to the right-hand side of (4), a term $\mathbf{N}_i\cdot\mathrm{curl}(\mathbf{M}_{\mathrm{FP}})$ occurs in Galerkin's equation, where the edge-shape functions are denoted by $\mathbf{N}_i$. This term can be rearranged into $\mathrm{curl}(\mathbf{N}_i)\cdot\mathbf{M}_{\mathrm{FP}}$, and therefore, $\mathrm{curl}(\mathbf{M}_{\mathrm{FP}})$ need not be calculated.

Solving (5) iteratively, the vector $\mathbf{f}$ can be determined from the previous solution $\mathbf{x}(t)$ using relation (2). So, the right-hand side became a function of time only

$$\mathbf{S}\cdot\mathbf{x}(t)+\mathbf{M}\cdot\dot{\mathbf{x}}(t)=\mathbf{g}(t). \tag{6}$$

The vectors $\mathbf{x}$ and $\mathbf{g}$ are now approximated as complex Fourier-series with N harmonics

$$\begin{aligned}\mathbf{x}(t)&\cong\mathrm{Re}\left\{\mathbf{X}_0+\sum_{k=1}^{N}\mathbf{X}_k e^{jk\omega_o t}\right\}\\ \mathbf{g}(t)&\cong\mathrm{Re}\left\{\mathbf{G}_0+\sum_{k=1}^{N}\mathbf{G}_k e^{jk\omega_o t}\right\}\end{aligned} \tag{7}$$

where ω_0 is the angular frequency of the excitation. $\mathbf{X}_0$ and $\mathbf{G}_0$ are the dc-components and $\mathbf{X}_k$ and $\mathbf{G}_k$ are the complex amplitudes at frequency $k\omega_0$. The approximations of $\mathbf{x}$ and $\mathbf{g}$ together with (6) lead to $N+1$ linear decoupled equation systems

$$\begin{aligned}\mathbf{S}\cdot\mathbf{X}_0&=\mathbf{G}_0\\ (\mathbf{S}+jk\omega_0\mathbf{M})\cdot\mathbf{X}_k&=\mathbf{G}_k\\ k&=1,\ldots,N.\end{aligned} \tag{8}$$

The amplitudes are calculated from the time signals by Fourier-transform. This transformation is normally done by the so-called discrete fourier transform (DFT)

$$\begin{aligned}\mathbf{G}_0&=\frac{1}{M}\sum_{n=0}^{M-1}\mathbf{g}(nT)\\ \mathbf{G}_k&=\frac{2}{M}\sum_{n=0}^{M-1}\mathbf{g}(nT)e^{-j\frac{2\pi}{M}kn}\\ k&=1,\ldots,M-1.\end{aligned} \tag{9}$$

Here, M is the number of time values for $\mathbf{g}$ and T is the time increment.

III. Algorithm

Starting from an arbitrary value of $\mathbf{M}_{\mathrm{FP}}$ (usually zero), the time dependent vector $\mathbf{g}(t)$ on the right-hand side of (6) is determined. From this time function, the amplitudes $\mathbf{G}_k$ $(k=0,\ldots,N)$ are calculated using (9). The equation system (8) is then solved and the new solution $\mathbf{x}(t)$ is determined by the relationship (7). With this solution and relation (2), the new value of $\mathbf{g}(t)$ can be evaluated. The solution of (8) is determined again with the new values for $\mathbf{G}_k$. This procedure will be repeated, until the mean and maximum variance of the reluctivity $(\nu_{\mathrm{mean}},\nu_{\max})$ over all integration points between the previous and actual iteration step is smaller than a prescribed value. The flow chart of the present method is shown in Fig. 1.

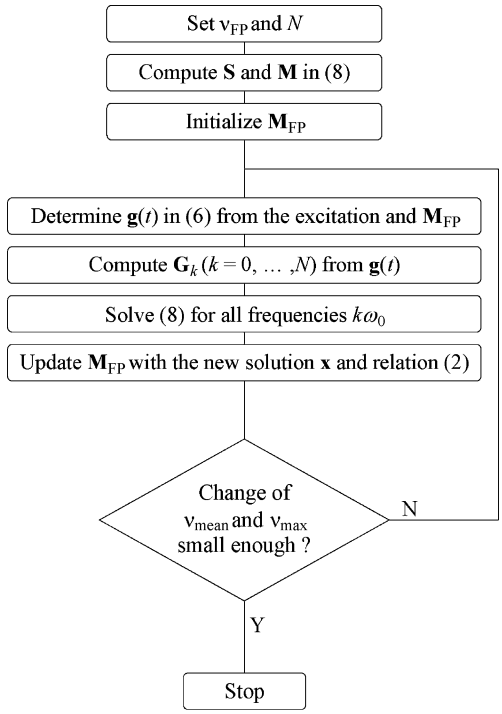

Fig. 1. Flow chart of the proposed method.

IV. Numerical Examples

A. Conductor Near Conducting Ferromagnetic Wall

The first 2-D problem consists of an aluminum conductor with a sinusoidal voltage per unit length of 0.66 V/m. Parallel to the aluminum conductor is a conducting ferromagnetic wall (see Fig. 2).

With the "brute force" method, six periods have to be stepped through until steady state is reached (see [2]), where each period was discretized in 20 time steps. The stopping criterion was chosen to be 0.1% for the mean relative variation and 1% for the maximum relative variation of the reluctivity in the integration points.

For the time domain method presented in [2], the same stopping criterion was used and the value of the relative permeability corresponding to ν_{FP} was taken to be 490.

With the method presented in this paper, two calculations with $N=3$ and $N=7$ harmonics have been made. For $N=3$, the relative permeability $\mu_{r,\mathrm{FP}}$ has been chosen to be 490 and for $N=7$ $\mu_{r,\mathrm{FP}}$ has been set to 150 to achieve faster convergence. The stopping criterions are the same as for the previous methods.

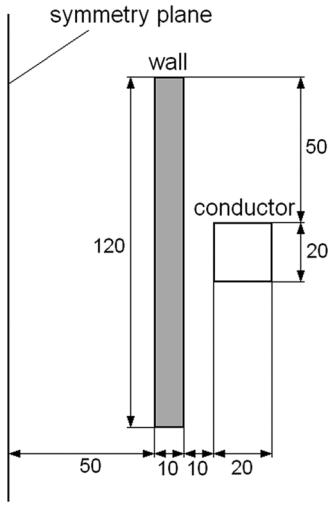

Fig. 2. Aluminum conductor with $\sigma = 3.5 \cdot 10^7$ S/m and a sinusoidal voltage of 0.66-V/m peak, 50 Hz. Ferromagnetic steel wall with $\sigma = 1.0 \cdot 10^7$ S/m.

TABLE I
NUMBER OF EQUATION SYSTEMS TO BE SOLVED

method	parameters	# of equations
brute force [2]	-	1,139
fixed point, time domain [2]	$\nu_{FP} = 490$	210
harmonic balance	$N = 3$ $\nu_{FP} = 490$	96
harmonic balance	$N = 7$ $\nu_{FP} = 150$	399

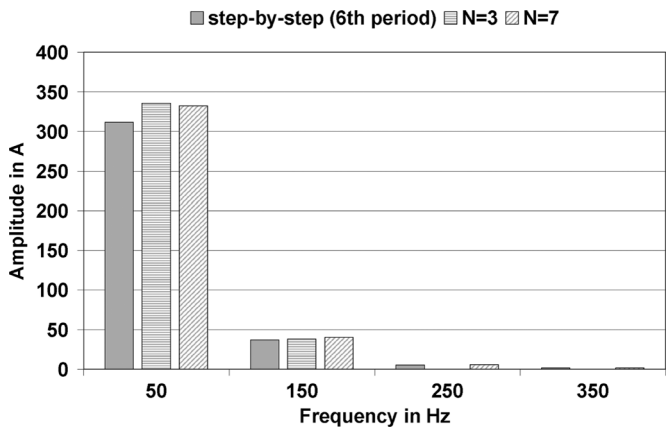

Fig. 3. Spectra of the total current in the ferromagnetic wall.

The three methods are compared in Table I. It can be seen, that the more harmonics are used, the more equation systems had to be solved. So, if it is *a priori* known that certain harmonics cannot occur, the number of equations can be reduced. If three harmonics are used, only 96 equations had to be solved, which is about 12 times lower then for the "brute force" method. For seven harmonics, the number of equation systems to be solved is about three times lower.

The spectra of the total current in the ferromagnetic wall obtained by various methods are shown in Fig. 3. It turned out that only the first (50 Hz) and third harmonic (150 Hz) are dominant. The amplitudes at 50 Hz and 150 Hz does not change a lot, if the number of harmonics taken into account rise from $N = 3$ to $N = 7$ (see Fig. 3). Hence, for the present method only the first and third harmonic had to be considered to achieve accurate results.

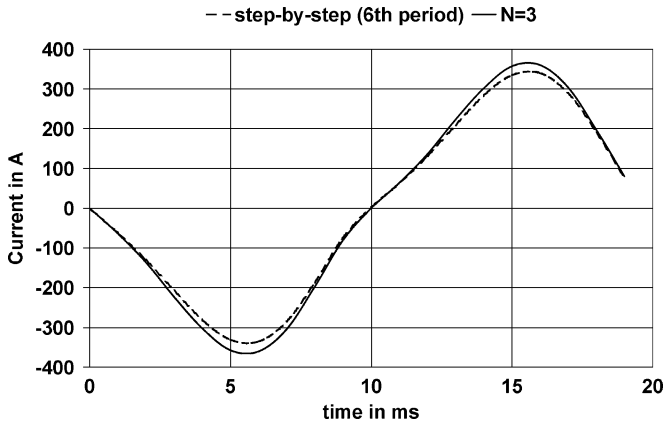

Fig. 4. Comparison of the total current in the ferromagnetic wall by present method and by step-by-step method.

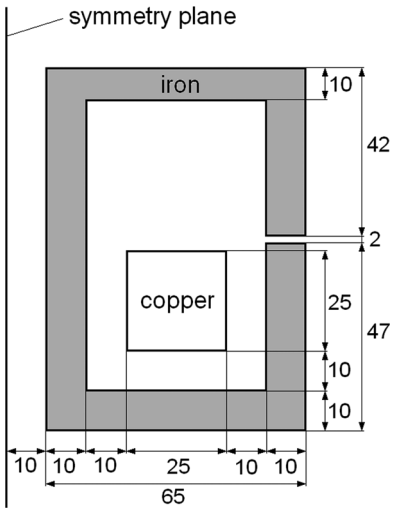

Fig. 5. Copper conductor ($\sigma = 5.7 \cdot 10^7$ S/m) driven by a sinusoidal voltage per unit length of 1.4-V/m peak and 50 Hz. The conductivity of the ferromagnetic screen is $\sigma = 1.0 \cdot 10^6$ S/m.

In Fig. 4, the sixth period of the current obtained by the step-by-step method is compared with the current obtained by the present method. Although only three harmonics are used for the Fourier-decomposition, the two curves show a good agreement. It turned out, that the applied time-stepping method strongly depends on the time increment T. Therefore, the agreement between the two curves can be improved, if the time increment for the time-stepping method is decreased.

B. Conductor Shielded by Conducting Ferromagnetic Screen

The second example consists of a copper conductor within a conducting ferromagnetic screen. The copper conductor is driven by a sinusoidal voltage. The dimensions are given in Fig. 5.

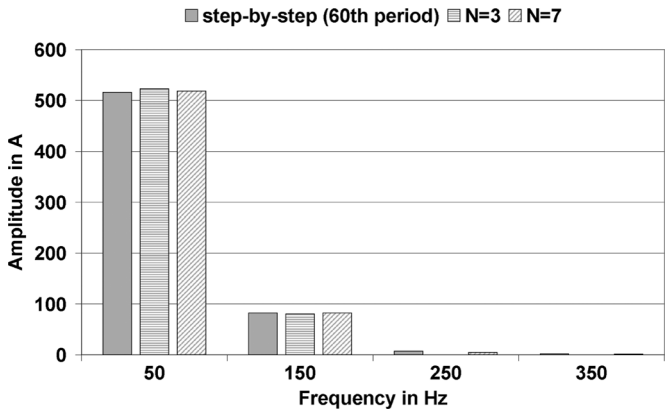

Fig. 6. Spectra of the total current in the copper conductor.

TABLE II
NUMBER OF EQUATION SYSTEMS TO BE SOLVED

method	parameters	# of equations
brute force [2]	-	19,058
fixed point, time domain [2]	$\nu_{FP} = 450$	980
harmonic balance	$N = 3$ $\nu_{FP} = 450$	87
harmonic balance	$N = 7$ $\nu_{FP} = 450$	231

The "brute force" method has required 60 periods to practically arrive at steady state [2], where each period has been discretized in 20 time steps. As in the example before, the stopping criterions are 0.1% for the mean and 1% for the maximum relative variance of the reluctivity.

For the time domain fixed-point method and for the harmonic balance method the relative permeability $\mu_{r,\mathrm{FP}}$ is taken to be 450. Again, the present method has been carried out for three and seven harmonics.

The spectra of the total current in the conductor are shown in Fig. 6. Again, the first and third harmonic are dominant. The agreement of the amplitude at 50 Hz between the time-stepping method and the presented method is better than in the previous example. The amplitudes of the first and third harmonic component slightly vary with increasing the number of harmonics. Hence, only three harmonics are necessary for the Fourier decomposition in (7).

The number of equation systems to be solved is shown in Table II. Due to the slower transient process, about 220 times $(N = 3)$ and about 83 times $(N = 7)$ more equations had to be solved with the step-by-step method than with the presented method. In this example, the convergence is better than in the previous example and, therefore, less iteration steps are needed.

V. CONCLUSION

The fixed-point method together with the harmonic balance method allows to linearize the finite-element Galerkin equations. For weak nonlinearities, just a few harmonics are necessary. If it is known *a priori* that certain harmonics cannot occur, the number of equations to be solved can be further reduced. Another advantage is, that the matrices $\mathbf{S}$ and $\mathbf{M}$ must be determined only once.

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