Over the mortar finite element method in linear electrostatics

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Abstract The present paper introduces a possible implementation of the mortar finite element method in electrostatics. It presents the equations of the linear electrostatics, and the main features of the mortar finite element method. Within the frame of the mortar finite element method one can couple different discretization schemes, the method being famous over its flexibility. In order to facilitate an easier way to couple the discretization schemes involved we will take advantage of a dual Lagrange multiplier space .The mortar finite element method is used to solve a plane problem in the context of linear electrostatics. An algorithm for implementing numerically this method is also proposed. Finally, a numerical example is discussed and conclusions are drawn.

AMS subject classifications: 65N15, 65N30, 65N55

1. Introduction

In this paper we present a possible implementation of the mortar finite element method in linear electrostatics. The mortar finite element method is a non-conforming finite element technique developed relatively recently, which provides a flexible algorithm for coupling different discretization schemes. One of the main features of the method is to replace the exact continuity condition at the skeleton of the decomposition with a weak one. Thus, this method is extremely useful in the study of domains with singularities such as cracks [4], or concave domains with corners. The method was proposed by C. Bernardi, et al. [1], [2]. Later on, B. Wohlmuth developed a saddle-point formulation of a certain problem within the frame of the mortar finite element method in [7]. The major improvement to the method in [7] was the use of a dual Lagrange multiplier space which permits the implementation of a much simpler matching between the triangulations. The same idea of implementing a dual Lagrange multiplier space is also implemented in the present paper, but in a more direct way without taking advantage of the explicit implementation of the Lagrange multiplicators. In section 2 there is a short presentation of the basic equations of linear electrostatics under certain assumptions. Section 3 presents the weak form of the problem stated in the second section and the features of the mortar finite element method within the frame of linear electrostatics, underlining its capabilities as a domain decomposition technique. Section 4 presents a possible numerical implementation, proposing a numerical algorithm.

2. Basic equations

Let D be a bounded convex domain in the bi-dimensional Euclidian space. Suppose that the domain D is occupied by an isotropic and homogenous medium. As in [3], the basic equations of linear electrostatics are:

- Gauss Law:

$$div\mathbf{D} = \rho \tag{2.1}$$

- Electrostatic form of the Faraday law:

$$rot\mathbf{E} = \mathbf{0} \tag{2.2}$$

- Constitutive relation:

$$\mathbf{D} = \varepsilon\mathbf{E} \tag{2.3}$$

where $\mathbf{E}$ is the electric field intensity and $\mathbf{D}$ is the electric flux density or, alternatively the electric displacement, ε is the electrical permittivity and ρ represents charge density.

We can see from (2.2) that $\mathbf{E}$ is a potential field, i.e. there exists an electric potential V (voltage) defined by:

$$\mathbf{E} = -\nabla V \tag{2.4}$$

In this way, supposing that ε is constant, from the relations (2.1)-(2.4) we can write:

$$-\varepsilon\Delta V = \rho \tag{2.5}$$

To this equation we'll attach the following boundary conditions:

$$V = \widetilde{V} \text{ on } \Gamma \tag{2.6}$$

where $\widetilde{V}$ are continuous functions given on Γ. Thus the boundary value problem is to find V which satisfy (2.5) and the boundary conditions (2.6).

In this way, by changing accordingly the variables, the first fundamental problem of linear electrostatics becomes:

$$-\Delta V = f \text{ on } D \tag{2.7}$$

$$V = 0 \text{ on } \Gamma. \tag{2.8}$$

where $f = \frac{\rho}{\varepsilon}$.

3. Weak formulation. The mortar finite element method

In order to implement the mortar finite element method, let's consider a non-overlapping decomposition of D into polyhedral subdomains D_k, $1 \le k \le K$,

$$\overline{D} = \bigcup_{k=1}^{K} \overline{D}_k \text{ where } D_l \bigcap D_k = \Phi, \forall l \neq k . \tag{3.1}$$

Let's suppose that each intersection $\partial D_l \bigcap \partial D_k$, $k \neq l$ is either empty, a vertex or a common edge. Each subdomain D_k is associated with a family of shape regular simplicial triangulations τ_{h_k}, $h_k \leq h_{k;0}$, where h_k is the maximum of the diameters of the elements in τ_{h_k}. Next, we'll follow the ideas from [7] and we consider linear conforming finite elements $S_1\left(D_k, \tau_{h_k}\right)$ on the individual subdomains, and enforce homogenous Dirichlet boundary conditions on $\Gamma \bigcap \Gamma_k$, where $\Gamma_k = \partial D_k$. Each common edge between two subdomains $\Gamma_k \bigcap \Gamma_l$, $k \neq l$, i.e. each interface is associated with an one dimensional triangulation, inherited from τ_{h_k} or τ_{h_l}. The above interface will be denoted by Γ_{kl} or Γ_{lk} if its triangulation is inherited from τ_{h_k} or from τ_{h_l}, respectively. In this way, we can write:

$$\Gamma_k = \bigcup_{l \in \overline{\mathcal{M}}(k)} \overline{\Gamma}_{kl} , \tag{3.2}$$

where $\overline{\mathcal{M}}(k)$ denotes the subset of $\{1,2,...,N\}$ so that $\Gamma_l \bigcap \Gamma_k$ is an interface for $l \in \overline{\mathcal{M}}(k)$. In the frame of the above, the union of all interfaces $\mathcal{S}$ can be decomposed uniquely:

$$\mathcal{S} = \bigcup_{k=1}^{K} \bigcup_{l \in \mathcal{M}(k)} \overline{\Gamma}_{lk} \tag{3.3}$$

Next we will introduce the mortars and the non-mortars, following the ideas from [1],[2] and[6]. We have $\mathcal{M}(k) \subset \overline{\mathcal{M}}(k)$ so that for each set $\{k,l\}$, $1 \leq k \leq K$, $l \in \overline{\mathcal{M}}(k)$ either $l \in \mathcal{M}(k)$ or $k \in \mathcal{M}(l)$ but not both.

The elements of $\{\Gamma_{kl} \mid 1 \leq k \leq K, l \in \mathcal{M}(k)\}$ are called mortars and the elements of $\{\Gamma_{lk} \mid 1 \leq k \leq K, l \in \mathcal{M}(k)\}$ are called non-mortars. The second set will be the one from which the Lagrange multiplier space, that we are about to introduce, will take its mesh. The Lagrange multiplier space will play a major role in the further development of the mortar finite element method.

In order to obtain the weak form of the first fundamental problem of linear electrostatics we define the following solutions space Ω:

$$\Omega = \left\{V \mid V \in H^1(D), V = 0 \text{ pe } \partial D\right\}, \tag{3.4}$$

We suppose that the solution is continuous on $\overline{D}$ and thus we'll have: $\Omega \equiv H_0^1(D)$. In this case the existence and uniqueness of the weak solution is an immediate consequence of the Lax-Milgram theorem applied on the space Ω.

We consider a weight function $W \in H_0^1(D)$. Multiplying the equation (2.7) by W and integrating the resulting equation on D, we can state the weak form of the first fundamental problem of linear electrostatics: let's determine $w \in \Omega$ so that:

$$\int_D W \Delta V d\Omega = \int_D V f d\Omega, \forall w \in H_0^1(D). \tag{3.5}$$

Let's define the following functionals on $H_0^1(D) \times H_0^1(D)$, and respectively on $H_0^1(D)$:

$$B(W,V)=\int_D (w_{,x}V_{,x}+w_{,y}V_{,y})dxdy \tag{3.6}$$

$$l(W)=\int_D Wfdxdy\,. \tag{3.7}$$

In this way, the weak form of the problem (2.14), (2.15) can be stated: find $V\in\Omega$, so that:

$$B(W,V)=l(W)\,,\forall W\in H_0^1(D). \tag{3.8}$$

One can notice immediately that the theory presented here for the Poisson equation still holds for any elliptical equation. In order to obtain the non-conforming aproximating weak form of the first fundamental problem of linear elasticity, we will define the following constrained aproximating solutions space Ω_h:

$$\Omega_h=\left\{W\,|\,W\in\prod_{k=1}^{K}S_1\left(D_k,\tau_{h_k}\right),b(W,\Theta)=0\ \text{pe}\ \partial D,\Theta\in\widetilde{\mathrm{M}}_h\right\}, \tag{3.9}$$

where $S_1\left(D_k,\tau_{h_k}\right)$ is the space of P_1 conforming finite elements and $\widetilde{\mathrm{M}}_h$ is the Lagrange multiplier space that will be specified in the following. In the definition (3.9), $b(\cdot,\cdot)$ is a bilinear functional so that:

$$b(W,\Theta)=\sum_{k=1}^{K}\sum_{l\in\mathcal{M}(k)}\left\langle[w_i],\mu_i\right\rangle_{\Gamma_{lk}},\ W\in\prod_{k=1}^{K}H^1(D_k),\ \Theta\in\prod_{k=1}^{K}\prod_{l\in\mathcal{M}(k)}\left(H^{\frac{1}{2}}(\Gamma_{lk})\right)', \tag{3.10}$$

where $[W]|_{\Gamma_{lk}}=W|_{\Gamma_l}-W_i|_{\Gamma_k}$ and $\langle\cdot,\cdot\rangle_{\Gamma_{lk}}$ is the duality pairing on $H^{\frac{1}{2}}(\Gamma_{lk})$. Within these notations, the non-conforming formulation of the mortar finite element method can be stated: find $V_h\in\Omega_h$ so that:

$$B(W_h,V_h)=l(W_h)\,,\forall W_h\in\Omega_h\,. \tag{3.11}$$

In (3.9), the Lagrange multiplier space $\widetilde{\mathrm{M}}_h$ is given by:

$$\widetilde{\mathrm{M}}_h=\prod_{k=1}^{K}\prod_{l\in\mathcal{M}(k)}\widetilde{\mathrm{M}}_h(\Gamma_{lk}), \tag{3.12}$$

where the local space $\widetilde{\mathrm{M}}_h(\Gamma_{lk})$ is a modified trace space of the finite element functions in $S_1\left(D_l,\tau_{h_l}\right)$. More details about these spaces can be found in [1], [2]. These spaces should satisfy the following properties:

$$\dim\widetilde{\mathrm{M}}_h(\Gamma_{lk})=\dim\left(H_0^1(\Gamma_{lk})\bigcap W_h(\Gamma_{lk})\right), \tag{3.13}$$

$$\widetilde{\mathrm{M}}_h(\Gamma_{lk})\subset W_h(\Gamma_{lk}), \tag{3.14}$$

where $W_h(\Gamma_{lk})$ is the trace space of $S_1(D_l, \tau_{h_l})$, i.e.

$$W_h(\Gamma_{lk}) = \left\{ W \in C^0(\Gamma_{lk}) \,|\, W = V|_{\Gamma_{lk}}, V \in S_1(D_l, \tau_{h_l}) \right\}. \quad (3.15)$$

As in [1],[2], $\widetilde{\mathrm{M}}_h(\Gamma_{lk})$ is found to be a subspace of $W_h(\Gamma_{lk})$ given by:

$$\widetilde{\mathrm{M}}_h(\Gamma_{lk}) = \left\{ W \in C^0(\Gamma_{lk}) \,|\, W = V|_{\Gamma_{lk}}, V \in S_1(D_l, \tau_{h_l}), \right.$$

$$\left. W|_e \in P_0(e), e \text{ is an edge of } D_k \text{ which contains an endpoint of } \Gamma_{lk} \right\}. \quad (3.16)$$

The nodal basis functions $\{\Phi_i\}_{i=1}^{N_{lk}}$ of $\widetilde{\mathrm{M}}_h(\Gamma_{lk})$ associated with the interior vertices p_j, $1 \le j \le N_{lk}$ of Γ_{lk} are given by:

$$\Phi_i(p_j) = \delta_{ij}, \quad (3.17)$$

where $N_{lk} = \dim \widetilde{\mathrm{M}}(\Gamma_{lk})$.

Next we will introduce the dual Lagrange multiplier space. The idea of using such dual basis functions spaces was first implemented in the work of L.R. Scott and S. Zhang[5]. Later on, B. Wohlmuth implemented the idea considering the dual of the Lagrange multiplier space in a saddle point formulation of a certain variational problem[6]. Here we will follow the ideas from[6], regarding the construction of the dual Lagrange basis functions space. Let's consider an edge σ and a polinomial space $\widetilde{P}_1(\sigma)$ so that:

$$P_0(\sigma) \subset \widetilde{P}_1(\sigma) \subset P_1(\sigma) \quad (3.18)$$

and $\{\Phi_i^\sigma\}_{i=1}^N$, $N = 1,2$ be a basis, so that:

$$\int_\sigma \Phi_i^\sigma ds \neq 0. \quad (3.19)$$

Within these notations, let's define a dual basis $\{\Psi_i^\sigma\}_{i=1}^N$, $N = 1,2$, $\Psi_i^\sigma \in \widetilde{P}_1(\sigma)$ so that:

$$\int_\sigma \Phi_i^\sigma \Psi_j^\sigma ds = \delta_{ij} \int_\sigma \Phi_i^\sigma ds, 1 \le i, j \le N. \quad (3.20)$$

From (3.20) we can see that the new basis is well defined [6] and, more:

$$\sum_{i=1}^N \Psi_i^\sigma = 1. \quad (3.21)$$

One can immediately notice that each Ψ_i^σ can be written as a linear combination of the $\{\Phi_i^\sigma\}_{i=1}^N$, $1 \le i \le N$ solving a $N \times N$ system. So we can state that:

$$\widetilde{P}_1(\sigma) = span\{\Psi_j^\sigma\}_{j=1}^N. \quad (3.22)$$

From the way of the introduction of Φ_i^σ we have:

$$\Phi_i = \sum_{\sigma \subset \mathrm{supp}\Phi_i} \Phi_i^\sigma , \tag{3.23}$$

where $\Phi_i^\sigma = \Phi_i |_\sigma$. Let's denote:

$$\widetilde{P}_1(\sigma) = \mathrm{span}\left\{\Phi_i^\sigma \mid 1 \le i \le N_{lk}, \sigma \subset \mathrm{supp}\Phi_i\right\}. \tag{3.24}$$

One can notice ([6]) that for $\partial\sigma \bigcap \Gamma_{lk} = \Phi$ we have:

$$\widetilde{P}_1(\sigma) = P_1(\sigma). \tag{3.25}$$

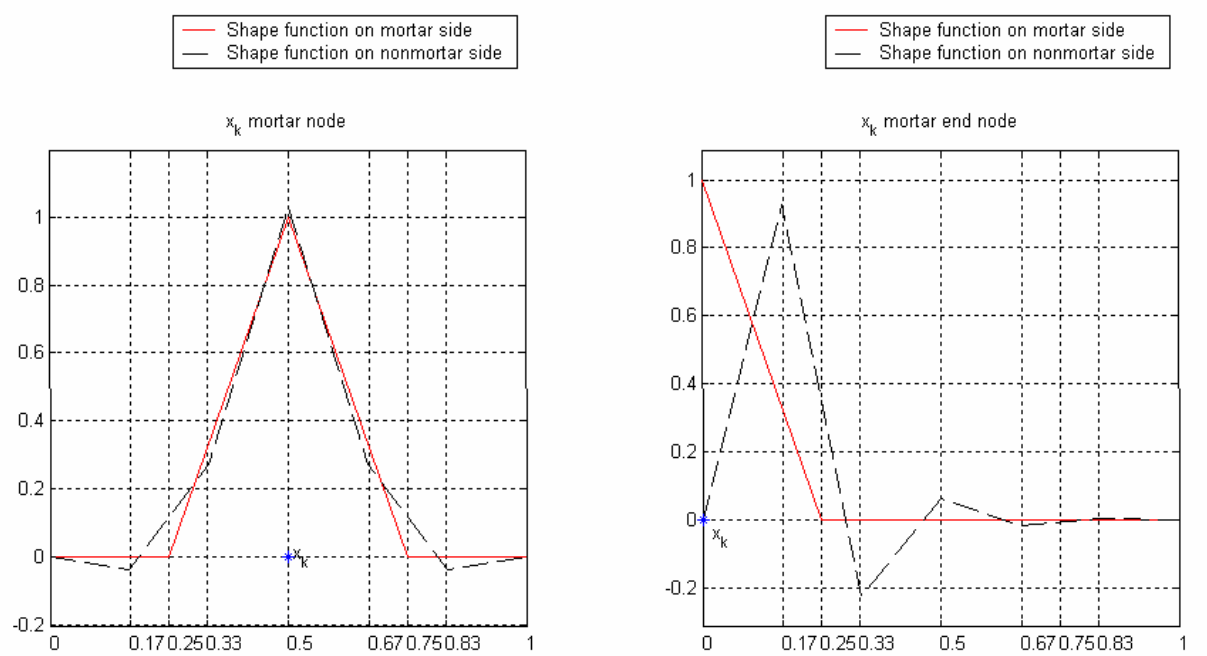

Figure 1 Mortar shape functions

Let's consider a global Lagrange multiplier space $\mathrm{M}_h(\Gamma_{lk})$ on every non-mortar $\Gamma_{lk}, 1 \le k \le K$, $l \in \mathcal{M}(k)$ so that:

$$\dim \mathrm{M}_h(\Gamma_{lk}) = \dim \widetilde{\mathrm{M}}_h(\Gamma_{lk}). \tag{3.26}$$

Within the frame of the notations previously introduced, the global basis functions of $\mathrm{M}_h(\Gamma_{lk})$ are given by:

$$\Psi_i = \sum_{\sigma \subset \mathrm{supp}\Phi_i} \Psi_i^\sigma . \tag{3.27}$$

A graphical representation of the dual basis functions is given in figure 1;usefull properties of the $\{\Psi_i\}_{i=1}^{N_{lk}}$ family are discussed in [7].

4. Numerical implementation

One can easily notice that the bilinear form $B(\cdot,\cdot)$ is uniformly elliptic on $\hat{\Omega}_h \times \hat{\Omega}_h$. This will imply that our problem (3.11) has a unique solution on $\hat{\Omega}_h$. Let's denote by $\mathbf{B}$ the matrix associated with the bilinear form $B(\cdot,\cdot)$ on $\hat{\Omega}_h \times \hat{\Omega}_h$ and by $\mathbf{l}$ the vector associated with the linear form $l(\cdot)$ on $\hat{\Omega}_h$, where $\hat{\Omega}_h$ is given by:

$$\hat{\Omega}_h = \left\{ w \mid w \in \prod_{k=1}^{K} S_1\left(D_k, \tau_{h_k}\right), b(W,\Theta) = 0 \text{ pe } \partial D, \Theta \in \mathrm{M}_h \right\}. \quad (4.1)$$

Thus the algebric form of the equation (3.11) is:

$$\mathbf{Bu} = \mathbf{l}. \quad (4.2)$$

A possible algorithm which implements this method can be the following one:

- define the model geometry
- define the non-overlapping subdomains
- generate the distinct triangulations on the subdomains
- assemble the matrix $\mathbf{B}$
- assemble the vector $\mathbf{l}$
- solve the system (4.2)
- postprocess the solution

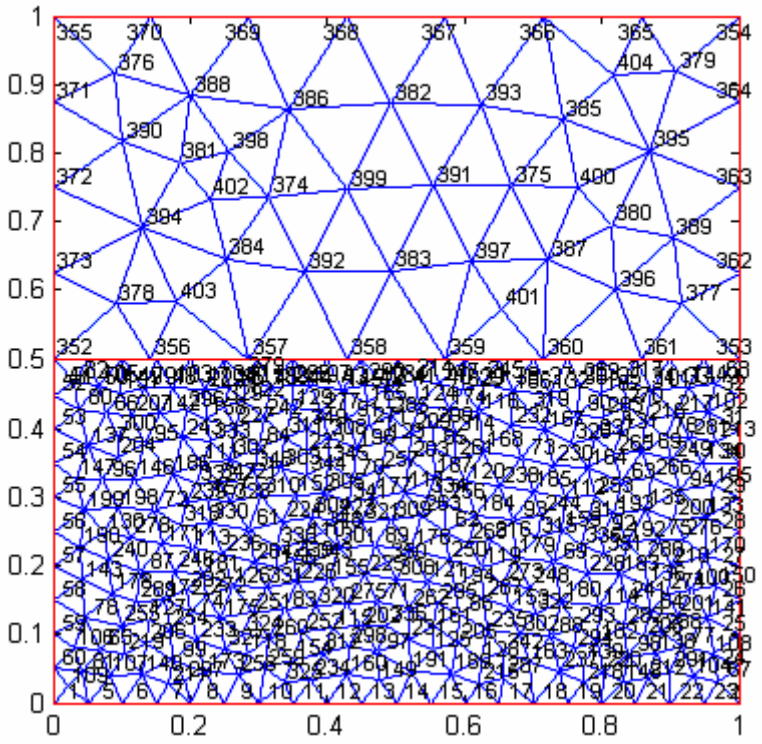

Figure 2 Non-matching grid

Let's consider our domain $D \equiv [0,1]\times[0,1]$ decomposed in two distinct subdomains $(0,1)\times(0,0.5)$ and $(0,1)\times(0.5,1)$ (figure2) with the associated unstructured triangulations of parameters 0.05 and 0.15.

In our numerical example we'll take the charge density ρ, so that the exact solution is given by:

$$v(x,y) = \sin \pi x \cdot \sin \pi y \,. \qquad (4.3)$$

In figure 3 (left) one can notice the mortar solution plotted against the analytical solution(right).

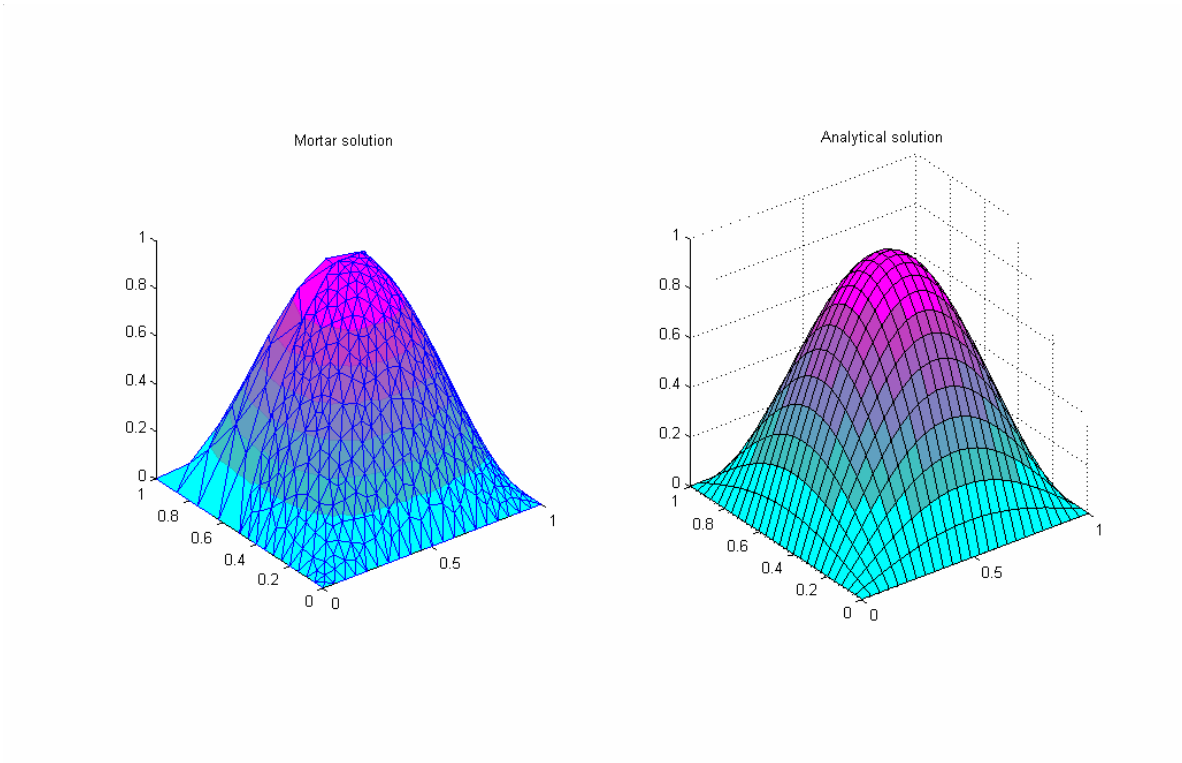

Figure 3 Mortar solution vs. analytical solution

In figure 4 there is a graphical representation of the error in the L2 norm versus the number of elements. From figure 4 we can see that there is a considerable increasing accuracy of the method as the number of elements increases.

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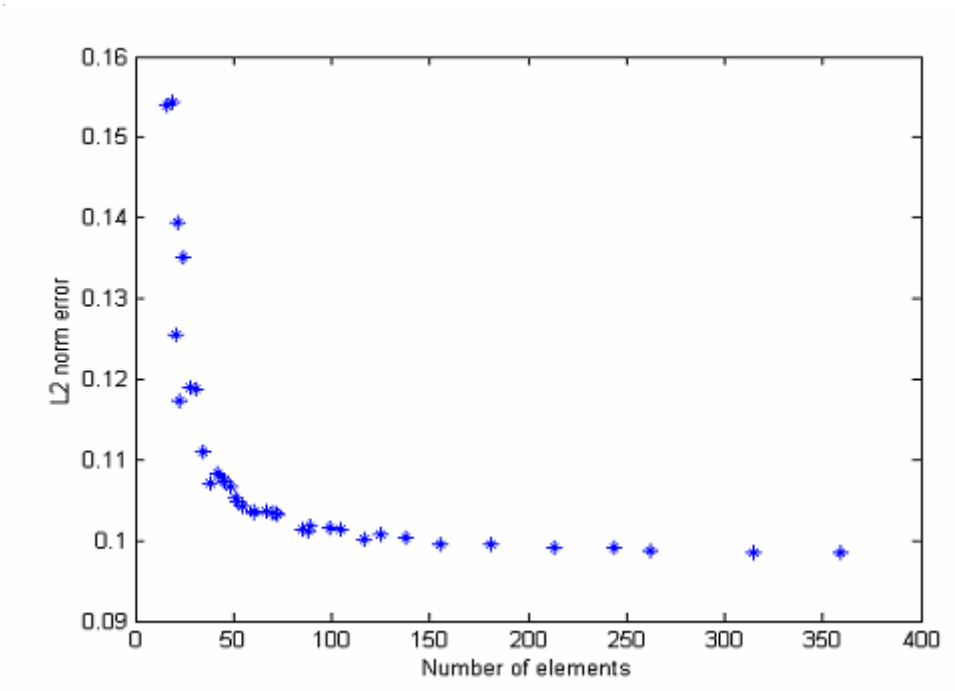

Figure 4 Error in L2 norm vs. number of elements

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