ERRORS IN NUMERICAL SOLVING OF NONLINEAR MAGNETIC FIELD PROBLEMS

Florea HANTILA, Mihai VASILIU, Mihai Maricaru
POLITEHNICA University of Bucharest, 060042, Bucharest, Romania, e-mail: hantila@elth.pub.ro
Costin IFRIM
Ecoair Corp., Hamden, CT 06517, USA, e-mail: cifrim@ecoair.com

Abstract. An effective technique to obtain a bound of the global error is presented. Nonlinear constitutive relationship is treated by the polarization iterative fixed point method (PFPM). Since the field problem to be solved at each iteration step is linear, by using the Green function method (GFM), an iterative integral expression of the error is obtained. PFPM being a Picard-Banach fixed point procedure, the bound for the norm of the difference between computed and exact solutions results as a sum of the error introduced in the iterative procedure and the error due to the chosen discretization mesh.

1 Introduction

Due to the complexity of the nonlinear electromagnetic field equations, numerical methods are the only option in the vast majority of problems. Numerical solutions represent, however, approximations of the exact field. Linear problems, in contrast to nonlinear ones, allow for a quantitative evaluation of the "distance" between the numerical solution and the exact one, e.g. static linear fields, where the hypercircle theorem [1], [2] applies.

Two main procedures are actually used for the evaluation of the computing errors. The first one takes into account discontinuities of the normal or tangential components induced by the numerical technique. This procedure is used particularly as a criterion for adaptive meshing. The second procedure is more consistent. It defines certain magnitudes that become zero when the numerical solution equals the exact one. These magnitudes – positively defined quantities – are called error estimators in what follows.

For static or stationary linear fields, according to the hypercircle principle, the "distance" is a suitable defined norm of the difference between the two solutions. For instance, let ($\mathbf{B}^*$,$\mathbf{H}^*$) be the exact solution of the boundary value field problem governed by $\nabla\cdot\mathbf{B}^*=\rho$, $\nabla\times\mathbf{H}^*=\mathbf{J}$ and $\mathbf{B}^*=\mu\mathbf{H}^*$, and let (**B**,**H**), be a set of independent fields, $\mathbf{B}\neq\mu\mathbf{H}$, solutions of $\nabla\cdot\mathbf{B}=\rho$ and $\nabla\times\mathbf{H}=\mathbf{J}$. The norm can be defined as

$$\left\|\mathbf{B}^* - \frac{\mathbf{B}+\mu\mathbf{H}}{2}\right\|_\nu = \frac{1}{2}\left\|\mathbf{B}-\mu\mathbf{H}\right\|_\nu \qquad \text{and} \qquad \left\|\mathbf{B}^*-\mathbf{B}\right\|_\nu \le \left\|\mathbf{B}-\mu\mathbf{H}\right\|_\nu \tag{1}$$

where $\|\mathbf{X}\|_\nu$ is the inner product $\|\mathbf{X}\|_\nu^2 = \int_\Omega \mathbf{X}\overline{\nu}\mathbf{X}d\Omega$, and $\overline{\nu}$, symmetric and positively defined, is the reluctivity tensor (inverse of the permeability tensor). Ω is the spatial domain of the field problem.

For nonlinear media a suitable error estimator is [3]:

$$L(\mathbf{B},\mathbf{H}) = W(\mathbf{B}) + W^*(\mathbf{H}) - \int_\Omega \mathbf{B}\cdot\mathbf{H}d\Omega \tag{2}$$

where $W(\mathbf{B})$ and $W^*(\mathbf{H})$ are the energy and the co-energy of the magnetic field. A similar error estimator is based on the constitutive relationship

$$\varepsilon = \left\|\mathbf{H} - F(\mathbf{B})\right\|_\mu \tag{3}$$

Relation (3) defines the error for the linear problem in the hypercircle principle sense.

The bound of the "distance" between numerical and exact solutions for nonlinear field problems, was reported in [4]. Both polarization iterative fixed point method (PFPM) and finite element

Paper supported by CERES project No: 64/2002 and Ecoair Corp.

method (FEM) were employed. In this paper PFPM and Green function method (GFM) are used to obtain this bound for the nonlinear field problem in unbounded regions.

2 Errors Evaluated by the PFPM Procedure

The stationary magnetic field verifies

$$\nabla \times \mathbf{H} = \mathbf{J}\,,\ \nabla \mathbf{B} = 0\,,\ \mathbf{H} = F(\mathbf{B}) \qquad \text{in } \Omega \tag{4}$$

In PFPM, the constitutive relationship $\mathbf{H} = F(\mathbf{B})$ is replaced by:

$$\mathbf{B} = \mu \mathbf{H} + \mathbf{I} \tag{5}$$

where the non-linearity is hidden in the polarization **I** [5]:

$$\mathbf{I} = \mathbf{B} - \mu F(\mathbf{B}) \equiv G(\mathbf{B})\ . \tag{6}$$

The permeability μ can be chosen such that the function $\boldsymbol{G}$ defined by (6) is a contraction, i.e.

$$\left\| G(\mathbf{B}') - G(\mathbf{B}'') \right\|_{\nu} \le \theta \left\| \mathbf{B}' - \mathbf{B}'' \right\|_{\nu}\,, \text{ for any } \mathbf{B}', \mathbf{B}'' \tag{7}$$

where θ <1. In the case of an isotropic medium one can choose at any point $P \in \Omega$, $\mu(P) < 2\,\mu_{min}(P)$ where:

$$\frac{1}{\mu_{min}} = \sup_{\mathbf{B}',\mathbf{B}''} \frac{\left| F(\mathbf{B}') - F(\mathbf{B}'') \right|}{\left| \mathbf{B}' - \mathbf{B}'' \right|} \qquad \frac{1}{\mu_{max}} = \inf_{\mathbf{B}',\mathbf{B}''} \frac{\left| F(\mathbf{B}') - F(\mathbf{B}'') \right|}{\left| \mathbf{B}' - \mathbf{B}'' \right|}\ . \tag{8}$$

Since μ_{min} is greater then the vacuum permeability μ_0, μ in (5) and (6) can be chosen to be μ_0. In this case $\theta = 1 - \dfrac{\mu_0}{\mu_M}$, where $\mu_M = \sup\limits_{P \in \Omega} \mu_{max}(P)$.

The PFPM consists in the following iterative procedure:

a) let $\mathbf{B}^{(0)}$ be an arbitrary initial value thus, from (6), $\mathbf{I}^{(0)} = G(\mathbf{B}^{(0)})$;

b) at each step *n*, $n \ge 1$, $\mathbf{B}^{(n)}$ and $\mathbf{H}^{(n)}$ are computed from the linear equations

$$\nabla \times \mathbf{H}^{(n)} = \mathbf{J}\,,\ \nabla \cdot \mathbf{B}^{(n)} = 0\,,\ \mathbf{B}^{(n)} = \mu\, \mathbf{H}^{(n)} + \mathbf{I}^{(n-1)} \tag{9}$$

c) the new polarization $\mathbf{I}^{(n)}$ is corrected by (6) i.e. $\mathbf{I}^{(n)} = G(\mathbf{B}^{(n)})$.

$\left(\mathbf{B}^{(n)}\right)_{n\ge1}$ and $\left(\mathbf{H}^{(n)}\right)_{n\ge1}$ are Picard-Banach sequences converging to the exact nonlinear solution ($\mathbf{B}^*$, $\mathbf{H}^*$). For simplicity we denote ($\mathbf{B}^{(1)}, \mathbf{H}^{(1)}$) of the first iteration by (**B, H**) – the exact linear solution of (9), and $\mathbf{I}^{(0)}$, by **I**. The deviation of the first value **B** and the target $\mathbf{B}^*$ is determined by:

$$\left\| \mathbf{B}^* - \mathbf{B} \right\|_{\nu} \le \frac{1}{1-\theta} \left\| \Delta \mathbf{I}^{(1)} \right\|_{\nu} \tag{10}$$

where $\left\| \Delta \mathbf{I}^{(1)} \right\|_{\nu} = \left\| \mathbf{I}^{(1)} - \mathbf{I} \right\|_{\nu}$. From the initial value **I,** determined from $\mathbf{B}^{(0)}$, **B** may be obtained through any numerical method, for example Newton-Raphson, combined with FEM or with GFM for unbounded regions. We need only one step in PFPM procedure (*n*=1) in order to evaluate the error with formula (10). The numerical solution $\mathbf{B}_{ap} = Z_{ap}(\mathbf{I})$ is an approximate value of **B,** the exact solution of the linear problem (9). From (6), we have $\mathbf{I}_{ap}^{(1)} = G(\mathbf{B}_{ap})$ and

$$\left\| \Delta \mathbf{I}^{(1)} \right\|_{\nu} = \left\| \mathbf{I}^{(1)} - \mathbf{I}_{ap}^{(1)} + \mathbf{I}_{ap}^{(1)} - \mathbf{I} \right\|_{\nu} \le \left\| G(\mathbf{B}) - G(\mathbf{B}_{ap}) \right\|_{\nu} + \left\| \mathbf{I}_{ap}^{(1)} - \mathbf{I} \right\|_{\nu} \le \theta \left\| \mathbf{B} - \mathbf{B}_{ap} \right\|_{\nu} + \left\| \mathbf{I}_{ap}^{(1)} - \mathbf{I} \right\|_{\nu}$$

Equation (10) becomes:

$$\left\| \mathbf{B}^* - \mathbf{B}_{ap} \right\|_{\nu} \le \left\| \mathbf{B}^* - \mathbf{B} \right\|_{\nu} + \left\| \mathbf{B} - \mathbf{B}_{ap} \right\|_{\nu} \le \frac{1}{1-\theta} \left\| \mathbf{B} - \mathbf{B}_{ap} \right\|_{\nu} + \left\| \mathbf{I}_{ap}^{(1)} - \mathbf{I} \right\|_{\nu} \tag{11}$$

The error in formula (11) can be expressed as

$$\left\|\mathbf{B}^{*}-\mathbf{B}_{ap}\right\|_{\nu}\leq\frac{1}{1-\theta}(\varepsilon_{Z_{ap}}+\varepsilon_{N}) \tag{12}$$

where $\varepsilon_{Z_{ap}}=\left\|\mathbf{B}-\mathbf{B}_{ap}\right\|_{\nu}$ is due to the approximation introduced by the numerical method used for solving the linear field problem and $\varepsilon_{N}=\left\|\mathbf{I}_{ap}^{(1)}-\mathbf{I}\right\|_{\nu}$ is due to the nonlinear procedure.

3 Determination of the Error Due to GFM Application

The problem of solving the magnetic field having non-zero sources **J** and ρ may be easily replaced by a field problem where the sources are zero. The constitutive relationship must be adjusted accordingly, while keeping the bounds μ_{min},μ_{max}. So, for simplicity we will farther consider that $\nabla\mathbf{B}=0$ and $\nabla\times\mathbf{H}=\mathbf{0}$.

If the polarization **I** is corrected by the flux density **B**, then the permeability μ in (5) and (6) can be chosen to be the vacuum permeability μ_0 [5]. The linear field problem of each iteration may be solved by using the Green functions for unbounded regions: $G=\frac{1}{4\pi R}$ or $G=\frac{1}{2\pi}ln\frac{1}{R}$, in 3-D or 2-D problems, respectively. For the numerical solution, the region with nonlinear media D_f is discretised in n_f elements and the flux density **B** is approximated by the average values in each element ω_i, i=1,2,…, n_f. In the 2-D case, we have:

$$\mathbf{B}_{ap_i}=\tilde{\mathbf{B}}_i=\frac{1}{\sigma_i}\int_{\omega_i}\mathbf{B}\,d\Omega=-\frac{1}{\sigma_i}\oint_{\partial\omega_i}A d\mathbf{l} \tag{13}$$

where σ_i and $\partial\omega_i$ are the area and respectively the boundary of the element ω_i. The polarization is obtained from (6) and therefore it has also constant values $\mathbf{I}_k$ in each element ω_k. The magnetic vector potential A is therefore given by:

$$A=-\frac{1}{2\pi}\sum_{k=1}^{n_f}\mathbf{I}_k\oint_{\partial\omega_k}ln\frac{1}{R}d\mathbf{l}'_k \tag{14}$$

Accordingly, (13) becomes:

$$\tilde{\mathbf{B}}_i=\frac{1}{\sigma_i}\left(\frac{1}{2\pi}\sum_{k=1}^{n_f}\oint_{\partial\omega_i}\oint_{\partial\omega_k}ln\frac{1}{R}\left(d\mathbf{l}_i d\mathbf{l}'_k\right)\right)\mathbf{I}_k=\frac{1}{\sigma_i}\overline{\overline{\alpha}}_{ik}\mathbf{I}_k \tag{15}$$

The error due to the numerical method is:

$$\varepsilon_{Z_{ap}}=\left\|\mathbf{B}-\tilde{\mathbf{B}}\right\|_{\nu}^{2}=\left\|\mathbf{B}\right\|_{\nu}^{2}-2\left\langle\mathbf{B},\tilde{\mathbf{B}}\right\rangle_{\nu}+\left\|\tilde{\mathbf{B}}\right\|_{\nu}^{2} \tag{16}$$

where the inner product and the norm are only computed for the nonlinear region D_f. Since $\left\langle\mathbf{B},\tilde{\mathbf{B}}\right\rangle_{\nu}=\left\|\tilde{\mathbf{B}}\right\|_{\nu}^{2}$ and $\left\|\mathbf{B}\right\|_{\nu}^{2}=\left\langle\mathbf{B},\mu\mathbf{H}+\mathbf{I}\right\rangle_{\nu}=\left\langle\mathbf{B},\mathbf{H}\right\rangle+\left\langle\mathbf{B},\mathbf{I}\right\rangle_{\nu}=\left\langle\mathbf{B},\mathbf{H}\right\rangle+\left\langle\tilde{\mathbf{B}},\mathbf{I}\right\rangle_{\nu}$, equation (16) becomes:

$$\varepsilon_{Z_{ap}}=\left\langle\mathbf{B},\mathbf{H}\right\rangle+\left\langle\tilde{\mathbf{B}},\mathbf{I}-\tilde{\mathbf{B}}\right\rangle_{\nu}=\left\langle\mathbf{B},\mathbf{H}\right\rangle-\left\langle\tilde{\mathbf{B}},\tilde{\mathbf{H}}\right\rangle=-\oint_{\partial D_f}V\mathbf{B}\cdot\mathbf{n}dS-\left\langle\tilde{\mathbf{B}},\tilde{\mathbf{H}}\right\rangle \tag{17}$$

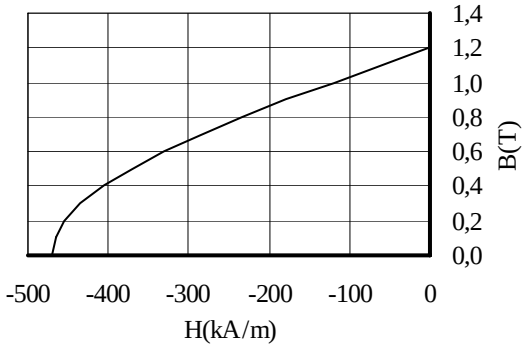

Fig.1. B-H relationship for PM

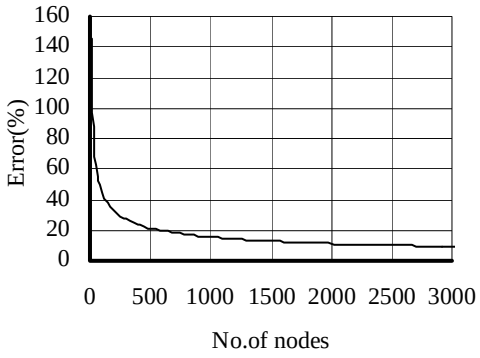

Fig.2. Global relative error

4 Numerical Examples

We consider a rectangular permanent magnet (PM) which has a B-H characteristic corresponding to the magnetization direction as plotted in Fig.1, with $\mu_{r_{max}}$ =16 and $\mu_{r_{min}}$ =1.2, and a transverse relative permeability μ_{r_T} =1.2. The PM is placed in 2-D unbounded space. The magnetic field is solved iteratively, using (15) and (6). The PM region is discretised in polygonal elements and so the tensors $\overline{\overline{\alpha}}_{ik}$ of equation (15) may be analyticaly calculated. The boundary integral needed in equation (17) is numericaly computed, using the scalar and vector potentials:

$$-\oint_{\partial D_f} V\mathbf{B}\cdot\mathbf{n}dS = -\oint_{\partial D_f} VdA = \cong -\frac{1}{2}\sum_k (V(P_{k+1})+V(P_k))(A(P_{k+1})-A(P_k)) \tag{18}$$

where the points P_k belong to the boundary D_f , vector potential *A* is analytically computed by equation (14) and scalar potential is also analytically computed with:

$$V = \frac{1}{2\pi}\sum_{k=1}^{n_f}\mathbf{I}_k \oint_{\partial\omega_k} ln\frac{1}{R}\mathbf{n}dl'_k \tag{19}$$

The corresponding global error in (12), normalised with respect to the remanent flux density, versus the number of nodes of the discretization mesh, is plotted in Fig.2.

5 Conclusions

A method for calculating a bound of the error of the numerical solution with respect to the exact solution, for nonlinear magnetic field problems, is presented. The non-linearity is treated by PFPM. In the cases of unbounded domains, GFM may be employed for solving the linear field problem of each iteration. The norm of the error due to GFM depends by the domain discretization, but it may be accurately computed (17), while the global error has a bound which depends strongly by the contraction factor (12) that may be in the proximity of the unit for strong nonlinear media. By applying the above presented techniques, the error due to the nonlinear iterative procedure can be easily brought arbitrary small.

References

[1] J.L.Synge, *The hypercircle in mathematical physics*, Cambridge Univ.Press, Cambridge, 1957.

[2] A.Bossavit, "A posteriori error bounds by "local corrections" using the dual mesh", *IEEE Trans. Magn.*, vol.35, no.3, pp. 1350-1353, 1999.

[3] J.Rikabi, C.F.Bryant, and E.M.Freeman, "An error-based approach to complementary formulations of static field solutions", *International Journal for Numerical Methods in Engineering*, vol.26, pp.1963-1987, 1988.

[4] I.R.Ciric, T.Maghiar, F.Hantila, and C.Ifrim, "Error Bounds for the FEM Numerical Solution of Nonlinear Field Problems", *EPNC'2002*, Leuven, (will be published in COMPEL).

[5] F.Hantila, G.Preda, and M.Vasiliu "Polarization method for static fields", *IEEE Trans. Magn.*, vol.36, no.4, pp. 672-675, 2000.