THEOREMS ABOUT THE MINIMUM OF THE POWER FUNCTIONAL IN LINEAR AND RESISTIVE ELECTRIC CIRCUITS

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Abstract. To determine the extreme of the power functional in case of the linear and resistive circuits is a problem of utmost importance, with quite useful theoretical and practical applications. In the present work it is demonstrated that the energetic steady state of the circuit, realized at a certain moment, represents a state\ theorems.

1. INTRODUCTION

Tellegen’s theorems have a special theoretic importance due to their generality and their help to easily demonstrate other practical important conclusions. Thus, for given steady state of a circuit, marked respectively with prime and second superscript, the line matrix of the voltages elements (branches) $[u]$, and the column matrix of the currents of the elements (branches) $[i]$, proves the relations, [1], [2]:

$$[u^{'}][i^{''}] = 0 \tag{1}$$

and

$$[u^{'}][i^{''}] - [u^{''}][i^{'}] = 0, \tag{2}$$

called respectively 1st and 2nd theorem of Tellegen.

From Tellegen’s 1st theorem, applied to the particular case when the two states are the same, we get the following relation between the voltage and current of a circuit element (branches) in a given steady state:

$$[u][i] = 0, \tag{3}$$

also called the power conservation theorem. If L is the number of elements (branches) of the circuit, while the voltage u_k, and the current i_k, of the each element are the same reference sense, we obtain the following relation, out of relation (3):

$$[u][i] = \sum_{k=1}^{L} u_k i_k = \sum_{k=1}^{L} p_k = 0, \tag{4}$$

which means that the algebrical sum of instantaneous absorbed powers at the terminals of the elements in a circuit is nill at any moment.

The same as with other conservative systems, such as mechanical or thermical [3], the electric conservative system in a steady state regime also represents an extreme energetic state [1], [4].

Because the theorem of the power conservation (4) does not show the energetic character of the electric circuit, the present work attempts a demonstration that the stationary and cvasistationary regime of the linear and resistive electric circuit represent a minimum state as for as the powers absorbed at the terminals elements of the circuit are concerned.

2. DETERMINING THE EXTREM OF THE POWER FUNCTIONAL FOR LINEAR AND RESISTIVE CIRCUIT

We take the case of a linear and resistive circuit in a stationary regime (d.c.). After transformation all the independent current source with equivalent independent voltages source, for each k branch of the L branches of the circuit, Ohm's theorem is a follows [5], (fig.1):

$$U_k = R_k I_k - E_k . \tag{5}$$

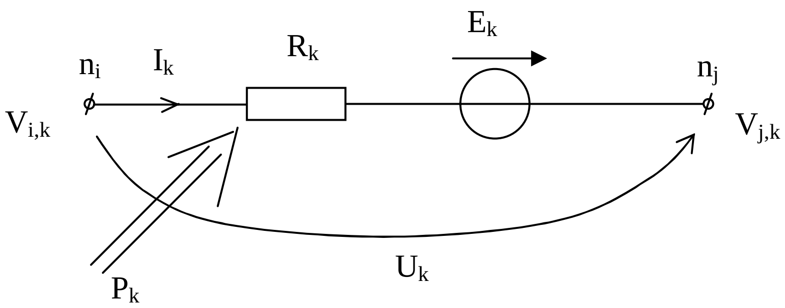

Fig. 1

If we mark the potentials of the nodes where the k branch is connected, $V_{i,k}$ and $V_{j,k}$, the current of each branch can be described using the 2nd Kirchhoff's theorem:

$$I_k = G_k (V_{i,k} - V_{j,k} + E_k) . \tag{6}$$

The absorbed power at all the branches of the circuit is described as functional

$$F \equiv P : R^N \to R,$$
$$F \equiv P(V_1, V_2, ..., V_N) = \sum_{k=1}^{L} R_k I_k^2 = \sum_{k=1}^{L} G_k (V_{i,k} - V_{j,k} + E_k)^2, \tag{7}$$

where $i, j = 1,2,..., N$, N being the number of nodes of the circuit.

The functional $F \equiv P(V_1, V_2, ..., V_N)$ is obviously a function of class C^2 within R^N set and it is positively defined, that $P(V_1, V_2, ..., V_N) \rangle 0, \forall \mathrm{V_i} \in R^N, i = 1,2,..., N.$ under this conditions, the extremes of the functional $F \equiv P(V_1, V_2, ..., V_N)$ are minimum points and they can be obtained by solving the system [6]:

$$\frac{\partial P}{\partial V_1} = 0,\ \frac{\partial P}{\partial V_2} = 0, ...,\ \frac{\partial P}{\partial V_N} = 0. \tag{8}$$

By using the power expression (7), the partial derivates of system (8) lead to the following equations system:

$$2\sum_{l_k \in n_1} I_k = 0,\ 2\sum_{l_k \in n_2} I_k = 0, ..., 2\sum_{l_k \in n_N} I_k = 0, \tag{9}$$

identical with the system of Kirchhoff equation for currents (1st Kirchhoff's theorem), expressed for all N nodes of circuit.

Consequently, we get the following theorem *(1st Theorem of Minimum Power – TMP1)*: *the minimum of the absorbed power by the branches of linear and resistive circuit in stationary regime (d.c.) is satisfied by the solutions in the currents and voltages of the circuit, and these are the currents and voltages which verify the 1st and 2nd theorem of Kirchhoff.*

This theorem has a general character and we can demonstrate a similar theorem for the cvasistationary regime (a.c.) of linear and resistive electric circuit.

By using the symbolical method, the voltage at every branch of the circuit is equal to:

$$\underline{U}_k = R_k \underline{I}_k - \underline{E}_k\ , \tag{10}$$

and the current of branch k can be expressed applying Kirchhoff's second theorem:

$$\underline{I}_k = G_k(\underline{V}_{i,k} - \underline{V}_{j,k} + \underline{E}_k). \tag{11}$$

The active power absorbed by all the L branches of the circuit is:

$$P = \sum_{k=1}^{L} R_k \left|\underline{I}_k\right|^2 = \sum_{k=1}^{L} G_k[(x_{i,k} - x_{j,k} + a_{E,k})^2 + (y_{i,k} - y_{j,k} + b_{E,k})^2], \tag{12}$$

where: the real and imaginary parts of the complex potential $\underline{V}_{i,k}$ and $\underline{V}_{j,k}$ of the nodes i and j where the k branches is connected, is

$$x_{i,k} = \mathrm{Re}[\underline{V}_{i,k}],\ y_{i,k} = \mathrm{Im}[\underline{V}_{i,k}], \tag{13}$$

$$x_{j,k} = \mathrm{Re}[\underline{V}_{j,k}],\ y_{j,k} = \mathrm{Im}[\underline{V}_{j,k}]\ , \tag{14}$$

and respectively, the real and imaginary parts of the independent voltage source of the k branch, is

$$a_{E,k} = \mathrm{Re}[\underline{E}_k], b_{E,k} = \mathrm{Im}[\underline{E}_k]\ . \tag{15}$$

The active power has been defined as the functional:

$$F \equiv P : R^{2N} \to R,$$

$$F \equiv P(x_1, x_2, ..., x_N, y_1, y_2, ..., y_N) = \sum_{k=1}^{L} G_k[(x_{i,k} - x_{j,k} + a_{E,k})^2 + (y_{i,k} - y_{j,k} + b_{E,k})^2], \tag{16}$$

and it is quite obviously a function class C^2 in R^{2N}, and is positively defined i.e. for all the pair $(x_i, y_i), i = 1,..., N,$ then $P(x_1, x_2,..., x_N, y_1, y_2,..., y_N)\rangle 0.$ Consequently, the minimum points of the active power functional is the solutions of the system [7], [8]:

$$\frac{\partial P}{\partial x_1} = 0,\ \frac{\partial P}{\partial y_1} = 0, \frac{\partial P}{\partial x_2} = 0,\ \frac{\partial P}{\partial y_2} = 0...,\ \frac{\partial P}{\partial x_N}, \frac{\partial P}{\partial y_N} = 0. \tag{17}$$

The partial derivates from the system (17) lead to the equations:

$$2 \sum_{l_k \in n_1} \underline{I}_k = 0,\ 2 \sum_{l_k \in n_2} \underline{I}_k = 0,..., 2 \sum_{l_k \in n_N} \underline{I}_k = 0, \tag{18}$$

which are identical to the Kirchhoff's equations for currents (1st Kirchhoff theorem), expressed for all the nodes N of the circuit.

Consequently, the following theorem can be issued *(2nd Theorem of Minimum Power –TMP 2)*: *the minimum of the active power absorbed by the branches of a linear and resistive circuit in a cvasistationary regime (a.c.) is satisfied by the solutions in currents and voltages of the circuit, and these are the currents and voltages that verify the 1st and 2nd theorem of Kirchhoff [9], [10].*

3. EXAMPLES

4.1. We consider the d.c. circuit shown in figure 2. The power absorbed of the branches of the circuit is (7):

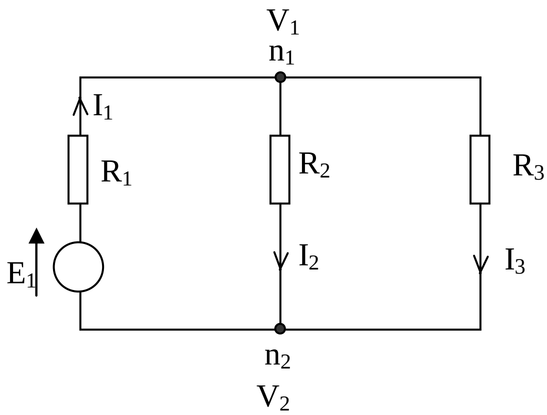

Fig. 2

$$P = G_1(V_2 - V_1 + E_1)^2 + G_2(V_1 - V_2)^2 + G_3(V_1 - V_2)^2.$$

The minimum of the absorbed power are solutions of the system (8), which represent the 1st theorem of Kirchhoff expressed in node 1 and 2:

$$\frac{\partial P}{\partial V_1} = -2G_1(V_2 - V_1 + E_1) + 2G_2(V_1 - V_2) + 2G_3(V_1 - V_2) = 2(-I_1 + I_2 + I_3) =$$
$$= 2 \sum_{l_k \in n_1} I_k = 0,$$

$$\frac{\partial P}{\partial V_2} = 2G_1(V_2 - V_1 + E_1) - 2G_2(V_1 - V_2) - 2G_3(V_1 - V_2) = 2(I_1 - I_2 - I_3) =$$
$$= 2 \sum_{l_k \in n_{21}} I_k = 0.$$

4.2. We consider the three-phase resistive circuit in star connection, shown in figure 3.

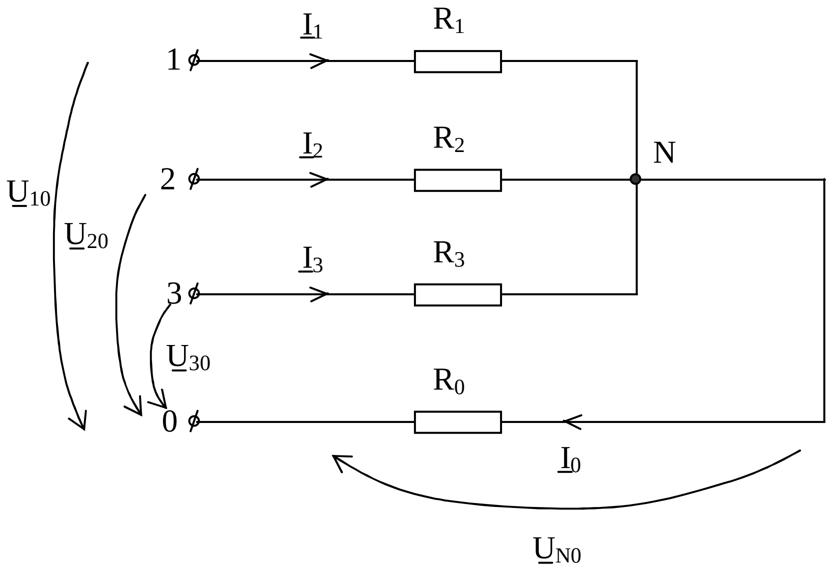

Fig. 3

If we know the three-phase symmetrical voltages of the system

$$\underline{U}_{10} = U, \underline{U}_{20} = U(-a - jb), \underline{U}_{30} = U(-a + jb),$$

and if we consider that the neutral point displacement voltage is

$$\underline{U}_{NO} = U(x + jy),$$

then the active absorbed power can be expressed by relation:

$$P = \sum_{k=1}^{3} R_k I_k^2 + R_0 I_0^2 = \sum_{k=1}^{3} G_k (U_{k0} - U_{N0})^2 + G_0 U_{N0}^2 =$$
$$= U^2 \{G_1[(1 - x^2) + y^2] + G_2[(-a - x^2) + (-b - y)^2] + G_3[(-a - x^2) + (b - y)^2] +$$
$$+ G_0(x^2 + y^2)\}.$$

The minimum of the active absorbed power are the solutions of the system:

$$\frac{\partial P}{\partial x} = U^2[-2G_1(1 - x) - 2G_2(-a - x) - 2G_3(-a - x) + 2G_0 x] = 0,$$
$$\frac{\partial P}{\partial y} = U^2[2G_1 y - 2G_2(-b - y) - 2G_3(-b - y) + 2G_0 y] = 0.$$

Results:

$$x=\frac{G_1-G_2a-G_3a}{\sum_{k=1}^{3}G_k+G_0},\; y=\frac{-G_2b+G_3b}{\sum_{k=1}^{3}G_k+G_0},$$

which are similarly with the formula: $\underline{U}_{N0}=\frac{\sum_{k=1}^{3}G_k\underline{U}_{k0}}{\sum_{k=1}^{3}G_k+G_0}$, obtained by using the 1st theorem of Kirchhoff in node N.

4. CONCLUSIONS

It has been established that the solutions of the linear and resistive electric circuit, in d.c. and a.c. regime, represent a minimum of the absorbed power in the circuit.

To find a satisfying answers to this problem, it is necessary to give an exact definitions of the power categories used in the a.c. periodic regime, written records of the specialists' agreement upon these definitions don't exist so far.

The energetic problem under debate in the present work has a wide range of practical applications and it aims at cutting down the wastes in the energetically systems.

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